HYPERBOLIC GROUPOIDS: METRIC AND MEASURE

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Abstract. We construct Patterson-Sullivan measure and a natural metric on the unit space of a hyperbolic groupoid. In particular, this gives a new approach to defining SRB measures on Smale spaces using Gromov hyperbolic graphs.

1. Introduction

In the previous paper [21] we gave the basic definitions related to the notion of a hyperbolic groupoid, and proved the duality theorem for them. Here we continue to study hyperbolic groupoids, and define natural classes of metrics and measures on their unit spaces.

Both constructions are classical in the particular cases of Gromov hyperbolic groups, Anosov flows, and Smale spaces. The metric is a generalization of what is sometimes called the *visual metric* on the boundary of the hyperbolic group (see [8, 7]). More on properties of this metric see [19]. In the case of Anosov flows

it is known as a "natural" or "dynamical" metric (on the stable and unstable leaves, or on the whole space). It was, for instance, defined by D. Fried in [6].

The definition of the metric depends on the choice of a *Busemann cocycle* and a number α close to zero. Two metrics associated with the same cocycle and α will be locally bi-Lipschitz equivalent, while metrics associated with different cocycles and numbers will belong to the same Hölder class.

The measure constructed here is equivalent to the Hausdorff measure for the metric, and is a direct generalization of the Patterson-Sullivan measure on the boundary of a hyperbolic group (see [22, 26, 4]). Applying this generalization to hyperbolic groupoids associated with Anosov flows and Smale spaces, we recover the classical Bowen-Margulis, or Sinai-Ruelle-Bowen measures. We get in this way a new approach to defining these measures: we represent the stable and unstable leaves as boundaries of Gromov hyperbolic graphs and then apply the Patterson-Sullivan construction.

Both the metric and the measure depend on the choice of a concrete *Busemann (quasi-)cocycle* ν on the groupoid of germs. The Busemann cocycle will play then the role of a logarithm of the derivative, in the sense that an element F of the pseudogroup multiplies the metric in a neighborhood of a point x roughly by $e^{-\alpha\nu(F,x)}$ and the measure by $e^{-\beta\nu(F,x)}$ for some positive numbers α and β. Here α is an arbitrary positive number close to zero (a parameter used in the construction of the metric), and β is the *entropy* of the groupoid (with respect to the cocycle ν).

Different choices of the cocycle may be natural in different situations. For example, by appropriate choices of the cocycle our construction gives either the measure of maximal entropy or the Hausdorff measure on the Julia set of a hyperbolic complex rational function (see Example 8.2).

For more on relations between hyperbolic geometry, Busemann cocycles, and conformal measures, see [14]. Relation between Gromov hyperbolicity and expanding dynamical systems (in particular from the metric point of view) is a subject of [9].

Overview of the paper. In the second section "Hyperbolic groupoids" we give a short overview of the notions developed in [21]. We remind the basic notations of the theory of pseudogroups and groupoids of germs, recall the notion of a log-scale, and review the main definitions related to hyperbolic groupoids and Smale quasi-flows.

Section "Dual Groupoid" gives a definition of the dual to a hyperbolic groupoid, in a slightly different way than it was done in [21]. We define a set of natural maps between cones of a Cayley graph of $\mathfrak{G}$, and then define the dual groupoid $\mathfrak{G}^\top$ as the groupoid of germs of the action of these maps on the boundary of $\mathfrak{G}$. This way we get approximations of the elements of the dual pseudogroup $\widetilde{\mathfrak{G}}^\top$ by partial maps on the set of vertices of the Cayley graph.

In Section 4 we study minimal hyperbolic groupoids. Minimality is a condition that simplifies many technicalities, though probably most of the results of this paper can be obtained in a more general setting. In particular, if a hyperbolic groupoid is minimal, then its geodesic flow is locally diagonal (Proposition 4.8), and we do not have to worry about this technical condition from [21].

We define the metric on the space of units of the groupoid in Section 5. Every Busemann cocycle $\nu : \mathfrak{G} \longrightarrow \mathbb{R}$ determines a natural log-scale on the boundary of the Cayley graph equal to the associated Gromov product. Its value $\ell(\xi_1, \xi_2)$ is

equal to minimum of the value of ν along a geodesic path connecting ξ_1 and ξ_2 in the Cayley graph of $\mathfrak{G}$. Using the Cayley graph of the dual groupoid $\mathfrak{G}^\top$ instead, we get a log-scale ℓ on the space of units of $\mathfrak{G}$. For any sufficiently small real number $\alpha > 0$ we can find a metric on $\mathfrak{G}^{(0)}$ such that $c^{-1}e^{-\alpha\ell(x,y)} \le |x-y| \le ce^{-\alpha\ell(x,y)}$ for some constant $c > 1$. We call such a metric *hyperbolic metric of exponent* α.

We show (Proposition 5.3) that then for every germ $g \in \mathfrak{G}$ there exists a neighborhood $U \in \widetilde{\mathfrak{G}}$ of g and a constant $c > 1$ such that

$$c^{-1}e^{-\alpha\nu(g)} \le \frac{|U(x) - U(y)|}{|x-y|} \le ce^{-\alpha\nu(g)}.$$

Moreover, we show in which sense this property characterizes the hyperbolic metrics, see Theorem 5.6. In particular, we show that if $\widetilde{\mathfrak{G}}$ acts by conformal maps on a compact subset of $\mathbb{C}$, then the usual metric on $\mathbb{C}$ is a hyperbolic metric of exponent 1 for $\mathfrak{G}$ with respect to the cocycle $\nu(F, x) = -\ln|F'(x)|$, see Proposition 5.7.

In "Growth and Entropy" we show that growth of cones (graded by the cocycle ν) in a Cayley graph of a hyperbolic groupoid is exponential, and give lower and upper estimates of the form $ce^{\beta n}$ on the growth, where β depends only on the pair $(\mathfrak{G}, \nu)$, and is called the *entropy* of the graded groupoid $(\mathfrak{G}, \nu)$. In fact, we prove more general estimates, which can be used to construct Gibbs measures for hyperbolic groupoids.

Patterson-Sullivan measures for hyperbolic groupoids are constructed in Section 7. It is a straightforward generalization of the classical construction, were we use the Cayley graphs of the dual groupoid $\mathfrak{G}^\top$ to construct the measure for the groupoid $\mathfrak{G}$. The measure μ is characterized by the property that the Radon-Nicodim derivative $\frac{dF_*\mu}{\mu}(x)$ is estimated from below and from above by functions of the form $ce^{-\beta\nu(F,x)}$, where β is the entropy of $(\mathfrak{G}, \nu)$. We also prove that the Patterson-Sullivan measure is equivalent to the Hausdorff measure of dimension β/α for the hyperbolic metric on $\mathfrak{G}^{(0)}$ of exponent α (see Corollary 7.5).

Note that in all these results the map $\nu : \mathfrak{G} \longrightarrow \mathbb{R}$ is a *quasi-cocycle*: the equality $\nu(g_1g_2) = \nu(g_1) + \nu(g_2)$ holds only up to an additive constant. In particular, we get only upper and lower estimates on the Radon-Nicodim derivative of the Patterson-Sullivan measure.

On the other hand, if $\nu : \mathfrak{G} \longrightarrow \mathbb{R}$ is *Hölder continuous* with respect to the hyperbolic metric on $\mathfrak{G}$ (a condition depending only on $\mathfrak{G}$ and ν), then our results can be made sharper. This case is analyzed in Section 8. We develop a duality theory for Hölder continuous cocycles, and show that in this case there exists a unique (up to a multiplicative constant) measure μ satisfying

$$\frac{dF_*\mu}{\mu}(x) = e^{-\beta\nu(F,x)}.$$

Moreover, in this case it is easy to construct an invariant measure on the geodesic flow of $\mathfrak{G}$. In all classical examples coming from Anosov flows, Smale spaces, and hyperbolic rational functions the natural cocycles are Hölder continuous, and our constructions produce the classical Sinai-Ruelle-Bowen and Bowen-Margulis measures.

2. Hyperbolic groupoids

2.1. **Groupoids and Pseudogroups.** Here we give a short review of notions related to pseudogroups and groupoids of germs.

A *pseudogroup* $\widetilde{\mathfrak{G}}$ acting on a space X is a collection of homeomorphisms between open subsets of X which is closed under:

- compositions;
- taking inverses;
- restricting onto open subsets;
- taking unions (if for a homeomorphism $F : U \longrightarrow V$ there exists a covering $\{U_i\}$ of U by open sets such that $F|_{U_i} \in \widetilde{\mathfrak{G}}$, then $F \in \widetilde{\mathfrak{G}}$).

We also assume that the identical homeomorphism $X \longrightarrow X$ belongs to $\widetilde{\mathfrak{G}}$.

A *germ* of an element of $\widetilde{\mathfrak{G}}$ is the equivalence class of a pair (F, x), where $F \in \widetilde{\mathfrak{G}}$, and x belongs to the domain of F. Here two pairs (F_1, x) and (F_2, x) are equivalent if there exists a neighborhood U of x such that $F_1|_U = F_2|_U$. The set of germs of $\widetilde{\mathfrak{G}}$ is a *groupoid*, i.e., it is a small category of isomorphisms with respect to the usual composition and taking inverses. We will denote the groupoid of germs of $\widetilde{\mathfrak{G}}$ by $\mathfrak{G}$. For a germ $g = (F, x) \in \mathfrak{G}$ we denote

$$\mathsf{o}(g) = x, \quad \mathsf{t}(g) = F(x).$$

Similarly, we will denote for $F \in \widetilde{\mathfrak{G}}$ by $\mathsf{o}(F)$ and $\mathsf{t}(F)$ the domain and range of F, respectively. Germs of the identity homeomorphism $X \longrightarrow X$ are called *units* of the groupoid and are identified with the corresponding points of X. We will also denote the set of units of a groupoid $\mathfrak{G}$ by $\mathfrak{G}^{(0)}$.

We use the following notation:

$$\mathfrak{G}_A = \{g \in \mathfrak{G} \ : \ \mathsf{o}(g) \in A\}, \quad \mathfrak{G}_B = \{g \in \mathfrak{G} \ : \ \mathsf{t}(g) \in B\},$$

and

$$\mathfrak{G}_A^B = \mathfrak{G}_A \cap \mathfrak{G}^B, \quad \mathfrak{G}|_A = \mathfrak{G}_A^A.$$

We also denote by $\mathfrak{G}^{(2)}$ the set $\{(g_1, g_2) \ : \ \mathsf{t}(g_2) = \mathsf{o}(g_1)\}$ of *composable pairs* of the groupoid $\mathfrak{G}$.

The groupoid of germs $\mathfrak{G}$ of a pseudogroup $\widetilde{\mathfrak{G}}$ has a natural topology. Namely, a basis of topology is given by the collection of open sets of the form

$$\{(F, x) \ : \ x \in \mathsf{o}(F)\}.$$

The groupoid $\mathfrak{G}$ is *topological* with respect to this topology, i.e., the multiplication $\mathfrak{G}^{(2)} \longrightarrow \mathfrak{G}$ and inversion $\mathfrak{G} \longrightarrow \mathfrak{G}$ are continuous.

On the other hand, the pseudogroup $\widetilde{\mathfrak{G}}$ is uniquely determined by the topological groupoid $\mathfrak{G}$. We say that $F \subset \mathfrak{G}$ is a *bisection* if $\mathsf{o} : F \longrightarrow \mathsf{o}(F)$ and $\mathsf{t} : F \longrightarrow \mathsf{t}(F)$ are homeomorphisms. Then every bisection F determines a homeomorphism from $\mathsf{o}(F)$ to $\mathsf{t}(F)$ by the rule $\mathsf{o}(g) \mapsto \mathsf{t}(g)$ for $g \in F$. It is easy to see that such homeomorphisms are elements of $\widetilde{\mathfrak{G}}$ and that every element $F \in \widetilde{\mathfrak{G}}$ defines a bisection $\{(F, x) \ : \ x \in \mathsf{o}(F)\}$. Hence, $\widetilde{\mathfrak{G}}$ is the pseudogroup of bisections of the groupoid of germs $\mathfrak{G}$. We will identify $F \in \widetilde{\mathfrak{G}}$ with the corresponding bisection (which is a subset of $\mathfrak{G}$). We will use therefore terminology of pseudogroups and groupoids as equivalent languages describing the same object.

Two units $x, y \in \mathfrak{G}^{(0)}$ belong to one $\mathfrak{G}$-*orbit* if there exists $g \in \mathfrak{G}$ such that $x = \mathsf{o}(g)$ and $y = \mathsf{t}(g)$. A subset $Y \subset \mathfrak{G}^{(0)}$ is said to be a $\mathfrak{G}$-*transversal* if it intersects every $\mathfrak{G}$-orbit.

Suppose that $f : Y \longrightarrow \mathfrak{G}^{(0)}$ is a local homeomorphism such that $f(Y)$ is a $\mathfrak{G}$-transversal. Then *localization* $\widetilde{\mathfrak{G}}|_f$ of $\widetilde{\mathfrak{G}}$ is the pseudogroup generated by all lifts by f of elements of $\widetilde{\mathfrak{G}}$, i.e., by homeomorphisms $F : U \longrightarrow V$ between open subsets

of Y such that $f|_U$ and $f|_V$ are homeomorphisms, and $f|_V \circ F \circ f|_U^{-1} \in \widetilde{\mathfrak{G}}$. We write the germs of the localization $\widetilde{\mathfrak{G}}|_f$ as triples (x, g, y), where $g \in \mathfrak{G}$, and $x, y \in Y$ are such that $f(x) = \mathsf{o}(g)$ and $f(y) = \mathsf{t}(g)$.

In particular, if $\mathcal{U} = \{U_i\}_{i \in \mathcal{I}}$ is an open covering of a $\mathfrak{G}$-transversal, then the corresponding localization $\mathfrak{G}|_{\mathcal{U}}$ consists triples $(i, g, j) \in \mathcal{I} \times \mathfrak{G} \times \mathcal{J}$ such that $\mathsf{o}(g) \in U_i$ and $\mathsf{t}(g) \in U_j$, which are multiplied by the rule

$$(i_1, g_1, j_1) \cdot (i_2, g_2, j_2) = (i_2, g_1 g_2, j_1),$$

where the product is defined if and only if $j_2 = i_1$ and $\mathsf{t}(g_2) = \mathsf{o}(g_1)$.

Definition 2.1. Two groupoids of germs $\mathfrak{G}_1, \mathfrak{G}_2$ are said to be *equivalent* if there exists a pseudogroup $\widetilde{\mathfrak{G}}$ acting on $\mathfrak{G}_1^{(0)} \sqcup \mathfrak{G}_2^{(0)}$ such that $\mathfrak{G}|_{\mathfrak{G}_1^{(0)}} = \mathfrak{G}_1$, $\mathfrak{G}|_{\mathfrak{G}_2^{(0)}} = \mathfrak{G}_2$, and every $\mathfrak{G}$-orbit is a union of one $\mathfrak{G}_1$-orbit and one $\mathfrak{G}_2$-orbit.

Two groupoids of germs are equivalent if and only if they have isomorphic localizations. (Just consider a collection of elements $F \in \widetilde{\mathfrak{G}}$, where $\widetilde{\mathfrak{G}}$ is as in the above definition, and $\mathsf{o}(F)$ and $\mathsf{t}(F)$ cover $\mathfrak{G}_1^{(0)}$ and $\mathfrak{G}_2^{(0)}$, respectively.)

A general definition of equivalence of topological groupoids (not only groupoids of germs) is a bit more complicated, see [21, 3.2.2] and references therein.

We will often deal with covers of compact subsets of $\mathfrak{G}$ by elements of $\widetilde{\mathfrak{G}}$. The following statement is proved in [21, Lemma 3.1].

Lemma 2.1. *Let $\widetilde{\mathfrak{G}}$ be a pseudogroup acting on a metric space. Let $C \subset \mathfrak{G}$ be a compact set, and let $\mathcal{U} \subset \widetilde{\mathfrak{G}}$ be a covering of C. Then there exists $\epsilon > 0$ such that for every $g \in C$ there exists $U \in \mathcal{U}$ such that $g \in U$ and the ϵ-neighborhood of $\mathsf{o}(g)$ is contained in $\mathsf{o}(U)$.*

If ϵ satisfies the conditions of the lemma for a covering $\mathcal{U}$, then we say that ϵ is a *Lebesgue number* of the covering. If $g \in U \in \widetilde{\mathfrak{G}}$ and the ϵ-neighborhood of $\mathsf{o}(g)$ is contained in $\mathsf{o}(U)$, then we say that g is *ϵ-contained* in U.

2.2. **Logarithmic scales.** It will be convenient sometimes to work with the following version of the notion of distance. For more details, see [21, Section 2.1].

Definition 2.2. A *log-scale* on a set X is a function $\ell : X \longrightarrow X \longrightarrow \mathbb{R} \cup \{+\infty\}$ such that

(1) $\ell(x, y) = \ell(y, x)$ for all $x, y \in X$;
(2) $\ell(x, y) = +\infty$ if and only if $x = y$;
(3) there exists $\delta > 0$ such that for any x, y, z we have

$$\ell(x, z) \geq \min(\ell(x, y), \ell(y, z)) - \delta.$$

It is proved in [21, Proposition 2.1] that for every log-scale and for all sufficiently small numbers $\alpha > 0$ there exists a metric $|x - y|$ on X and a number $C > 1$ such that

$$c^{-1} e^{-\alpha \ell(x,y)} \leq |x - y| \leq c e^{-\alpha \ell(x,y)}$$

for all $x, y \in X$. We say in this case that $|x - y|$ is an *associated metric of exponent α* for the log-scale. Note that any two metrics associated with ℓ are Hölder equivalent to each other. In particular, they define the same topology.

Accordingly to the definition of an associated metric, we say that a map f between two sets with log-scales is *Lipschitz* if there exists $c > 0$ such that $\ell(f(x), f(y)) \geq \ell(x, y) - c$ for all x, y. A map f is *bi-Lipschitz* if it is invertible, and the maps f

and f^{-1} are Lipschitz. A map f is *Hölder* if there exist constants $c > 1$ and $\eta > 0$ such that $\ell(f(x), f(y)) \geq c\ell(x, y) + \eta$ for all x, y.

Two log-scales ℓ_1, ℓ_2 on a set X are said to be *Lipschitz equivalent* if $|\ell_1(x, y) - \ell_2(x, y)|$ is uniformly bounded for all x, y such that $x \neq y$. They are *Hölder equivalent* if there exist constants $c > 1$ and $\eta > 0$ such that

$$c^{-1}\ell_1(x, y) - \eta \leq \ell_2(x, y) \leq c\ell_1(x, y) + \eta$$

for all x, y.

Let $\widetilde{\mathfrak{G}}$ be a pseudogroup acting on a space $\mathfrak{G}^{(0)}$. A *Lipschitz structure* on $\widetilde{\mathfrak{G}}$ (or on the corresponding groupoid of germs $\mathfrak{G}$) is a log-scale on $\mathfrak{G}^{(0)}$ such that every element of $\widetilde{\mathfrak{G}}$ is locally bi-Lipschitz (i.e., if every germ $g \in \mathfrak{G}$ has a bi-Lipschitz neighborhood $U \in \widetilde{\mathfrak{G}}$) with respect to the log-scale. More on Lipschitz structures on pseudogroups, see [21, Section 3.5].

We will use the following notations. Let F and G be two real-valued functions. We write $F \doteq G$ if the difference $|F - G|$ is uniformly bounded for all values of the variables. We will write $F \asymp G$ if there exists a constant $c > 1$ such that $c^{-1}F \leq G \leq cF$ for all values of the variables.

2.3. **Hyperbolic groupoids.** We will present here a review of the notions related to hyperbolic groupoids. For more details see [21]. We assume that the reader is familiar with the basic theory of Gromov-hyperbolic graphs (otherwise, see [21, Section 2.2] and the references therein).

We say that a subset X of the set of units of a groupoid $\mathfrak{G}$ is a *topological transversal* if X contains an open transversal of the groupoid.

Definition 2.3. A *generating pair* (S, X) of a groupoid $\mathfrak{G}$ is a compact subset $S \subset \mathfrak{G}$ and a compact topological transversal X such that for every $g \in \mathfrak{G}|_X$ there exists n such that the set $\bigcup_{0 \leq k \leq n}(S \cup S^{-1})^k$ is a neighborhood of g in $\mathfrak{G}|_X$.

If (S, X) is a generating pair of $\mathfrak{G}$ and $x \in X$, then the *Cayley graph* $\mathfrak{G}(x, S)$ is the directed graph with the set of vertices $\mathfrak{G}_x^X$ in which there is an arrow from g to h whenever there exists $s \in S$ such that $h = sg$.

Definition 2.4. A groupoid of germs $\mathfrak{G}$ is *hyperbolic* if there exists a compact generating pair (S, X), a metric $|\cdot|$ defined on a neighborhood of X, and numbers $\lambda \in (0, 1), \delta, \Lambda, \Delta > 0$ such that

(1) each element of $\widetilde{\mathfrak{G}}$ is locally Lipschitz;
(2) each element $g \in S$ is a germ of a λ-contraction $F \in \widetilde{\mathfrak{G}}$;
(3) $\mathsf{o}(S) = \mathsf{t}(S) = X$;
(4) for every $x \in X$ the Cayley graph $\mathfrak{G}(x, S)$ is δ-hyperbolic;
(5) for every $x \in X$ there exists $\omega_x \in \partial\mathfrak{G}(x, S)$ such that every directed path in the Cayley graph $\mathfrak{G}(x, S^{-1})$ is a (Δ, Λ)-quasigeodesic converging to ω_x.

Here a (Δ, Λ)-*quasi-geodesic* is a (finite or infinite) sequence $v_0, v_1, \ldots,$ of vertices such that $|v_i - v_{i+1}| < \Delta$ for all i, (where $|v_i - v_{i+1}|$ is the combinatorial distance in the graph) and $|i - j| \leq \Lambda|v_i - v_j|$ for every pair of indices i, j.

The *Busemann cocycle* β_ω on a Gromov-hyperbolic graph Γ, where $\omega \in \partial\Gamma$, is given by

$$\beta_\omega(v_1, v_2) = \lim_{v \to \omega}(|v_1 - v| - |v_2 - v|),$$

where v_1, v_2 are vertices of the graph, $|\cdot|$ is the combinatorial metric on the graph (the smallest number of edges in a path connecting the vertices), and we choose any one of the partial limits on the right-hand side. The number β_ω is uniquely defined, up to an additive constant (which depends only on δ).

Definition 2.5. An *η-quasi-cocycle* on a groupoid $\mathfrak{G}$ is a map $\nu : \mathfrak{G}|_X \longrightarrow \mathbb{R}$, where X is a topological transversal, such that

(1) for every $g \in \mathfrak{G}|_X$ there exists a neighborhood U of g such that $|\nu(g) - \nu(h)| < \eta$ for all $h \in U \cap \mathfrak{G}_X$;
(2) $|\nu(g_1g_2) - \nu(g_1) - \nu(g_2)| < \eta$ for any composable pair g_1, g_2 of elements of $\mathfrak{G}|_X$.

A *graded* groupoid is a groupoid $\mathfrak{G}$ together with a quasi-cocycle. Two quasi-cocycles $\nu_1 : \mathfrak{G}|_{X_1} \longrightarrow \mathbb{R}$ and $\nu_2 : \mathfrak{G}|_{X_2} \longrightarrow \mathbb{R}$ define the same grading (are *strongly equivalent*) if there exists a quasi-cocycle $\nu : \mathfrak{G}|_{X_1 \sqcup X_2} \longrightarrow \mathbb{R}$ such that $|\nu(g) - \nu_1(g)|$ and $|\nu(h) - \nu_2(h)|$ are bounded (where $g \in \mathfrak{G}|_{X_1}$ and $h \in \mathfrak{G}|_{X_2}$).

Two quasi-cocycles $\nu_1, \nu_2 : \mathfrak{G}|_X \longrightarrow \mathbb{R}$ are *coarsely equivalent* if there exist constants $\Lambda > 1$ and $c > 0$ such that

$$\Lambda^{-1}\nu_1(g) - c \le \nu_2(g) \le \Lambda\nu_1(g) + c$$

for all $g \in \mathfrak{G}|_X$.

One can check that the Busemann cocycle $\beta_{\omega_x}(g_1, g_2)$ on the Cayley graph $\mathfrak{G}(x, S)$ depends (up to an additive constant) only on $g_1g_2^{-1}$. We get thus a quasi-cocycle $\nu(g_1g_2^{-1}) = \beta_{\omega_x}(g_1, g_2)$, which we will also call a *Busemann quasi-cocycle* on the groupoid $\mathfrak{G}$. More generally, any quasi-cocycle that is coarsely equivalent to ν will be called a *Busemann quasi-cocycle* of the groupoid $\mathfrak{G}$. One can show that any two Busemann quasi-cocycles (defined by different generating pairs) are coarsely equivalent to each other (see [21, Proposition 4.11]).

A *graded hyperbolic groupoid* $(\mathfrak{G}, \nu)$ is a hyperbolic groupoid together with a strong equivalence class of a Busemann quasi-cocycle.

Definition 2.6. We say that a subset $C \subset \mathfrak{G}$ of a graded hyperbolic groupoid $(\mathfrak{G}, \nu)$ is *positive* if for every $g \in C$ we have $\nu(g) > 2\eta$, where η is as in Definition 2.5.

A subset $C \subset \mathfrak{G}$ is *contracting* if for every $g \in C$ there exists a contracting map $U \in \widetilde{\mathfrak{G}}$ such that $g \in U$.

If C is positive, then for every composable product $\ldots g_2g_1$ of elements of C the path $g_1, g_2g_1, g_3g_2g_1, \ldots$ is a quasi-geodesic path converging to a point of $\partial\mathfrak{G}(x, S) \setminus \{\omega_x\}$.

We denote

$$\partial\mathfrak{G}_x = \partial\mathfrak{G}(x, S) \setminus \{\omega_x\},$$

where $\partial\mathfrak{G}(x, S)$ is the boundary of the hyperbolic graph $\mathfrak{G}(x, S)$. One can show that $\partial\mathfrak{G}(x, S)$, ω_x, and $\partial\mathfrak{G}_x$ do not depend on the choice of (S, X). Denote also by $\overline{\mathfrak{G}_x^X}$ the space $\mathfrak{G}_x^X \cup \partial\mathfrak{G}_x$, i.e., the completion of the Cayley graph $\mathfrak{G}(x, S)$ with the point ω_x removed. Note that $\overline{\mathfrak{G}_x^X}$ does not depend on S (but depends on X).

If (S, X) is a compact generating pair satisfying the conditions of Definition 2.4, then we denote

$$T_x = \bigcup_{n \ge 0} S^n \cap \mathfrak{G}_x,$$

i.e., T_x is the set of elements of $\mathfrak{G}_x$ representable as a product of elements of S (where inverses are not allowed).

Similarly, we denote $T_g = \bigcup_{n\geq 0} S^n \cdot g$ for $g \in \mathfrak{G}$. We obviously have a bijection $x \mapsto x \cdot g$ between $T_{\mathsf{t}(g)}$ and T_g.

We denote by $\mathcal{T}_g$ the intersection of the closure of T_g in $\overline{\mathfrak{G}_x^X}$ with $\partial\mathfrak{G}_x$ for $x = \mathsf{o}(g)$. It is equal to the set of points of $\partial\mathfrak{G}_x$ that can be represented as infinite products

$$\ldots g_2 g_1 g = \lim_{n\to\infty} g_n \cdots g_1 \cdot g$$

for $g_i \in S$. We will denote $\overline{T_g} = T_g \cup \mathcal{T}_g$.

The following proposition is proved in [21, Proposition 4.4].

Proposition 2.2. *Let $(\mathfrak{G}, \nu_0)$ be a graded hyperbolic groupoid. Let X be a compact topological $\mathfrak{G}$-transversal.*

Then there exist a compact generating set S of $\mathfrak{G}|_X$, a metric $|\cdot|$ on a neighborhood $\widehat{X}$ of X, and an η-quasi-cocycle $\nu : \mathfrak{G}|_{\widehat{X}} \longrightarrow \mathbb{Z}$ strongly equivalent to ν_0, such that

(1) *for every $g \in S$ we have $\nu(g) > 3\eta$;*
(2) $\mathsf{o}(S) = \mathsf{t}(S) = X$;
(3) *there exists $\lambda \in (0,1)$ such that for every $g \in S$ has a λ-contracting neighborhood $U \in \widetilde{\mathfrak{G}}|_{\widehat{X}}$;*
(4) *every element $g \in \mathfrak{G}|_X$ is equal to a product of the form $g_n \cdots g_1 \cdot (h_m \cdots h_1)^{-1}$ for some $g_i, h_i \in S$.*

The following proposition is proved in the same way as [21, Proposition 4.9].

Proposition 2.3. *Let (S, X) be a generating pair of $\mathfrak{G}$ satisfying the conditions of Proposition 2.2. Then there exists a compact set $A \subset \mathfrak{G}|_X$ such that for every $h \in \mathfrak{G}_X$ there exists $a \in A$ such that $\overline{T_{ah}}$ is a neighborhood of $\overline{T_h}$.*

2.4. **Smale quasi-flows.** We present here definition of the notion of a *Smale quasi-flow* generalizing the classical notion of a *Smale space* (see [25, 24]). More details can be found in [21, Sections 2.3, 3.5, 6].

Let R be a topological space. A *direct product structure* on R is given by a continuous map $[\cdot,\cdot] : R \times R \longrightarrow R$ such that

$$[x,x] = x, \quad [[x,y],z] = [x,z], \quad [x,[y,z]] = [x,z]$$

for all $x, y, z \in R$. One can show that if $[\cdot,\cdot]$ defines a direct product structure, then we can find topological spaces A, B and a homeomorphism $\pi : A \times B \longrightarrow R$ such that $[\pi(a_1, b_1), \pi(a_2, b_2)] = \pi(a_1, b_2)$. Namely, we can take $A = \mathsf{P}_1(R, x)$ and $B = \mathsf{P}_2(R, x)$, where

$$\mathsf{P}_1(R,x) = \{y \in R \,:\, [x,y] = x\}, \qquad \mathsf{P}_2(R,x) = \{y \in R \,:\, [x,y] = y\}, \tag{1}$$

and $\pi(y_1, y_2) = [y_1, y_2]$.

A *local product structure* on a space X is given by a covering (called an *atlas*) of X by open subsets R_i (called *rectangles*) together with a direct product structure $[\cdot,\cdot]_{R_i}$ on them such that for any pair R_i, R_j of rectangles and for any $t \in R_i \cup R_j$ there exists a neighborhood U of t and a direct product structure $[\cdot,\cdot]_U$ on it such that $[x,y]_{R_i} = [x,y]_U$ and $[x,y]_{R_j} = [x,y]_U$ whenever the corresponding expressions are defined.

If X is a space with a local product structure, then an open subset $R \subset X$ and a direct product structure $[\cdot,\cdot]_R$ on R is a *rectangle* of X if when we add it to an

atlas of X, we get again an atlas of X. In particular, we can define the maximal atlas of a local product structure consisting of all rectangles of X.

We say that a metric $|x-y|$ agrees with a local product structure on X if for every rectangle $R = A \times B$ of X there exist metrics $|\cdot|_A$ and $|\cdot|_B$ on A and B such that restriction of $|x-y|$ onto R is locally bi-Lipschitz equivalent to the metric

$$|(a_1, b_1) - (a_2, b_2)|_R = \max\{|a_1 - a_2|_A, |b_1 - b_2|_B\}.$$

A pseudogroup $\widetilde{\mathfrak{G}}$ acting on a space X with a local product structure *preserves the local product structure* if for any germ (F, z) of $\widetilde{\mathfrak{G}}$ there exist rectangles R_i and R_j such that $z \in R_i$, $F(z) \in R_j$ and $F([x, y]_{R_i}) = [F(x), F(y)]_{R_j}$ for all x, y belonging to a neighborhood of z. Note that if $\widetilde{\mathfrak{G}}$ preserves a local product structure then for any germ $g \in \mathfrak{G}$ of $\widetilde{\mathfrak{G}}$ there exist rectangles $R_{\mathsf{o}(g)} = A_{\mathsf{o}(g)} \times B_{\mathsf{o}(g)}$ and $R_{\mathsf{t}(g)} = A_{\mathsf{t}(g)} \times B_{\mathsf{t}(g)}$ and a neighborhood $F \in \widetilde{\mathfrak{G}}$ of g such that $\mathsf{o}(F) = R_{\mathsf{o}(g)}$, $\mathsf{t}(F) = R_{\mathsf{t}(g)}$, and the map $F : R_{\mathsf{o}(g)} \longrightarrow R_{\mathsf{t}(g)}$ can be decomposed into a direct product of maps $A_F : A_{\mathsf{o}(g)} \longrightarrow A_{\mathsf{t}(g)}$ and $B_F : B_{\mathsf{o}(g)} \longrightarrow B_{\mathsf{t}(g)}$. In particular, the groupoid of germs $\mathfrak{G}$ of $\widetilde{\mathfrak{G}}$ has a local product structure in a natural way. *Projections* of the germ g are the germs $\mathsf{P}_1(g)$ and $\mathsf{P}_2(g)$ of A_F and B_F, respectively, at the points $a \in A$ and $b \in B$ such that $(a, b) = \mathsf{o}(g)$. We also denote $A_F = \mathsf{P}_1(F)$ and $B_F = \mathsf{P}_2(F)$.

Let $\mathfrak{G}$ be a groupoid of germs preserving a local product structure on $\mathfrak{G}^{(0)}$, and let ν be a quasi-cocycle defined on a restriction of $\mathfrak{G}$ onto a compact topological transversal X. We say that ν agrees with the local product structure if there exists an open covering $\mathcal{R}$ of X by rectangles and a constant $c > 0$ such that if $\mathsf{P}_i(g_1) = \mathsf{P}_i(g_2)$ for some $g_1, g_2 \in \mathfrak{G}$ and $i \in \{1, 2\}$, then $|\nu(g_1) - \nu(g_2)| < c$.

A groupoid $\mathfrak{G}$ preserving a local product structure on $\mathfrak{G}^{(0)}$ is *locally diagonal* if there exits a covering $\mathcal{R}$ of a topological $\mathfrak{G}$-transversal by open rectangles such that if for $g \in \mathfrak{G}$ either $\mathsf{P}_1(g)$ or $\mathsf{P}_2(g)$ is a unit, then g is a unit.

Definition 2.7. A *Smale quasi-flow* is a groupoid $\mathfrak{G}$ together with an η-quasi-cocycle $\nu : \mathfrak{G}|_X \longrightarrow \mathbb{R}$ and a local product structure on $\mathfrak{G}^{(0)}$ such that there exists a compact generating pair (S, X), a metric $|\cdot|$ defined on a neighborhood of X, and a number $\lambda \in (0, 1)$ such that

1. (1) the metric $|\cdot|$ and the quasi-cocycle ν agree with the local product structure;
2. (2) $\widetilde{\mathfrak{G}}$ acts by locally Lipschitz transformations with respect to $|\cdot|$;
3. (3) $\mathsf{o}(S) = \mathsf{t}(S) = X$, and $\nu(g) > 3\eta$ for all $g \in S$;
4. (4) for every $g \in S$ there exists a rectangular neighborhood $F \in \widetilde{\mathfrak{G}}$ of g such that restrictions of F and F^{-1} onto $\mathsf{P}_1(\mathsf{o}(F), x)$ and $\mathsf{P}_2(\mathsf{o}(F), x)$, respectively, are λ-contractions for all $x \in \mathsf{o}(F)$;
5. (5) for every compact subset $C \subset X$ and for every real number $k > 0$ the closure of the set $\{g \in \mathfrak{G}|_X \ : \ |\nu(g)| \le k\}$ is compact;
6. (6) the groupoid $\mathfrak{G}$ is locally diagonal.

For definition of sets $\mathsf{P}_i(R, x)$, see (1). We denote P_1 and P_2 in the case of a Smale quasi-flow by P_+ and P_-, respectively.

Let $(\mathfrak{G}, \nu)$ be a Smale quasi-flow. Let $\mathcal{R}$ be a covering of a topological $\mathfrak{G}$-transversal by sufficiently small rectangles. Consider localization $\mathfrak{G}|_{\mathcal{R}}$ of $\mathfrak{G}$ onto $\mathcal{R}$, and denote by $\mathsf{P}_+(\mathfrak{G})$ and $\mathsf{P}_-(\mathfrak{G})$ the groupoids of germs of pseudogroups generated by projections $\mathsf{P}_+(F)$ and $\mathsf{P}_-(F)$ of rectangular elements of the pseudogroup

$\widetilde{\mathfrak{G}}|_{\mathcal{R}}$. It is proved in [21] that groupoids $\mathsf{P}_+(\mathfrak{G})$ and $\mathsf{P}_-(\mathfrak{G})$ are well defined up to equivalence of groupoids (i.e., their equivalence class does not depend on the choice of $\mathcal{R}$). They are called *Ruelle groupoids* of the quasi-flow $\mathfrak{G}$.

We will need the following technical result of [21, Proposition 6.3].

Proposition 2.4. *Every Smale quasi-flow is equivalent to a groupoid $\mathfrak{H}$ satisfying the following properties.*

The space of units $\mathfrak{H}^{(0)}$ is a disjoint union of a finite number of rectangles $W_1 = A_1 \times B_1, \ldots, W_n = A_m \times B_m$.

There exists an open transversal X_0 equal to the union of open sub-rectangles $W_i^\circ = A_i^\circ \times B_i^\circ \subset R_i$ such that the closure of R_i° is compact. Denote by X the union of closures of the rectangles W_i°.

There exists a finite set $\mathcal{S}$ of elements of the pseudogroup $\widetilde{\mathfrak{H}}$ such that

1. *every $F \in \mathcal{S}$ is a rectangle $A_F \times B_F = \mathsf{P}_+(F) \times \mathsf{P}_-(F)$;*
2. *for every $F \in \mathcal{S}$ there exist $i, j \in 1, \ldots, n$ such that $\mathsf{o}(F) \subset W_i$, $\mathsf{t}(F) \subset W_j$, $\mathsf{o}(A_F) = A_i$, $\mathsf{t}(B_F) = B_j$;*
3. *intersections of $\mathsf{o}(F)$ and $\mathsf{t}(F)$ with X are non-empty;*
4. *A_F and B_F^{-1} are λ-contracting for some $\lambda \in (0, 1)$;*
5. *$S = \{(F, x) : x, F(x) \in X\}$ is a generating set of $\mathfrak{H}|_X$ (i.e., (S, X) is a generating pair);*
6. *$\mathsf{o}(S) = \mathsf{t}(S) = X$;*
7. *$\nu(g) > 2\eta$ for all germs of elements of $\mathcal{S}$;*

3. Dual groupoid

Let $(\mathfrak{G}, \nu)$ be a graded hyperbolic groupoid. Let (S, X) be a generating pair of $\mathfrak{G}$ satisfying the conditions of Proposition 2.2 for the quasi-cocycle ν. Let $\mathcal{S}$ be a finite covering of S by contracting positive elements of $\widetilde{\mathfrak{G}}$.

Let $A \subset \mathfrak{G}$ be a compact set satisfying the conditions of Proposition 2.3. Suppose also that for any two sequences g_i, h_i of germs of elements of $\mathcal{S}$ an equality $\ldots g_2 g_1 \cdot g = \ldots h_2 h_1 \cdot h$ for some g, h, $\mathsf{o}(g) = \mathsf{o}(h) \in X$ implies that for all sufficiently big n there exists m and $a \in A$ such that $a g_n \cdots g_1 g = h_m \cdots h_1 h$. Existence of such a set A follows from hyperbolicity of the Cayley graphs of $\mathfrak{G}$ and the fact that all directed paths in $\mathfrak{G}(x, S)$ are quasi-geodesics.

Find then a finite covering $\mathcal{A} = \{U\}$ of A by bi-Lipschitz elements of $\widetilde{\mathfrak{G}}$. Let $\widehat{A}$ be the set of germs of the elements of $\mathcal{A}$.

The following lemma is proved in the same way as [21, Lemma 5.3].

Lemma 3.1. *Let ϵ be a common Lebesgue number of the coverings $\mathcal{S}$, $\mathcal{A}$, and $\mathcal{A}^{-1}$ of S, A, and A^{-1}, respectively. There exists $0 < \delta_0 < \epsilon$ such that the following condition is satisfied.*

Let U_i, V_i, $i = 1, 2, \ldots$ be finite or infinite sequences of elements of the set $\mathcal{S} \cup \mathcal{A}$ in which at most one element belongs to $\mathcal{A}$. Let $|x - y| < \delta_0$ for $x, y \in X$, the ϵ-neighborhoods of $U_i \cdots U_1(x)$ and $V_i \cdots V_1(x)$ are contained in $\mathsf{o}(U_{i+1})$ and $\mathsf{o}(V_{i+1})$, respectively. Then an equality

$$(\ldots U_2 U_1, x) = (\ldots V_2 V_1, x)$$

of finite or infinite products of germs implies

$$(\ldots U_2 U_1, y) = (\ldots V_2 V_1, y).$$

Fix δ_0 satisfying the conditions of Lemma 3.1. Suppose that $g \in \mathfrak{G}|_X$ and $h \in \mathfrak{G}$ are such that $|\mathsf{t}(g) - \mathsf{t}(h)| < \delta_0$. For a finite or infinite product $\xi = \ldots g_2 g_1 g \in \overline{T_g}$, where $g_i \in S$, find elements $U_i \in \mathcal{S}$ such that g_i is ϵ-contained in U_i. Define then

$$R_g^h(\xi) = \ldots U_2 U_1 \cdot h. \tag{2}$$

By Lemma 3.1, $R_g^h(\xi)$ depends only on g, h, and ξ (and does not depend on the choice of the generators g_i or the choice of the elements U_i). Note that $R_g^h(\xi) \notin \mathfrak{G}|_X$ in general (even for $\xi \in \mathfrak{G}_x^X$).

For every $h \in \mathfrak{G}$ we have a natural homeomorphism $\xi \mapsto \xi \cdot h$ from $\partial\mathfrak{G}_{\mathsf{t}(h)}$ to $\partial\mathfrak{G}_{\mathsf{o}(h)}$ defined by

$$\ldots g_2 g_1 \cdot g \mapsto \ldots g_2 g_1 \cdot gh.$$

Note that every germ of this homeomorphism is also a germ of a transformation of the form R_g^{gh} for some $g \in \mathfrak{G}_{\mathsf{t}(h)}$.

The proof of the following proposition is straightforward.

Proposition 3.2. *Let $g \in \mathfrak{G}_X$ and $h \in \mathfrak{G}$ are such that R_h^g is defined. Then the map R_h^g is bi-Lipschitz with respect to the log-scale defined by the Gromov product. Moreover, there exists a constant $c > 0$ (not depending on g and h) such that*

$$\ell(\xi_1, \xi_2) + \nu(g) - \nu(h) - c \le \ell(R_h^g(\xi_1), R_h^g(\xi_2)) \le \ell(\xi_1, \xi_2)) + \nu(g) - \nu(h) + c$$

for all $\xi_1, \xi_2 \in \overline{T_h}$. It particular, R_h^g is a homeomorphism between $\mathcal{T}_h$ and $R_h^g(\mathcal{T}_h)$.

Here ℓ is defined using a generating set of $\mathfrak{G}|_{X_2}$ where X_2 contains $\mathsf{o}(\mathcal{S}) \cup \mathsf{t}(\mathcal{S})$. Note that different choices of the log-scale are bi-Lipschitz equivalent to each other.

In [21, Theorem 5.1] it was proved that the disjoint union $\partial\mathfrak{G} = \bigcup_{x \in \mathfrak{G}^{(0)}} \partial\mathfrak{G}_x$ has a natural topology and a local products structure coming from the maps R_g^h. Namely, if $U \in \widetilde{\mathfrak{G}}$ is a sufficiently small neighborhood of an element $g \in \mathfrak{G}$, and ξ is an interior point of $\mathcal{T}_g$, then a neighborhood of ξ in $\partial\mathfrak{G}$ is the set

$$R_{g,U} = \bigcup_{h \in U} R_g^h(\mathcal{T}_g^\circ),$$

where $\mathcal{T}_g^\circ$ is the interior of $\mathcal{T}_g \subset \mathfrak{G}_{\mathsf{o}(g)}$. The set $R_{g,U}$ is naturally homeomorphic to $\mathsf{o}(U) \times \mathcal{T}_g^\circ$, where the homeomorphism is given by the map

$$(x, \zeta) \mapsto [x, \zeta]_U := R_g^{(U,x)}(\zeta). \tag{3}$$

These direct product decompositions agree with each other, and we get in this way a local product structure and topology on $\partial\mathfrak{G}$. For more details, see [21, Section 5].

Every $U \in \widetilde{\mathfrak{G}}$ defines a local homeomorphism of $\partial\mathfrak{G}$ with domain $\partial\mathfrak{G}_{\mathsf{t}(U)}$ and range $\partial\mathfrak{G}_{\mathsf{o}(U)}$, mapping $\xi \in \partial\mathfrak{G}_{U(x)}$ for $x \in \mathsf{t}(U)$ to $\xi \cdot (U, x)$. The groupoid of germs of the pseudogroup generated by such maps is called the *geodesic quasi-flow* of $\mathfrak{G}$. Its elements can be written as pairs (ξ, g), where $\xi \in \partial\mathfrak{G}_{\mathsf{t}(g)}$, $\mathsf{o}(\xi, g) = \xi$, and $\mathsf{t}(\xi, g) = \xi \cdot g$. We denote the geodesic flow by $\partial\mathfrak{G} \rtimes \mathfrak{G}$.

Theorem 3.3. *The space $\mathfrak{d}\mathfrak{G}_X$ of germs of restrictions of the maps R_g^h, for $g \in \mathfrak{G}|_X, h \in \mathfrak{G}$ onto open subsets of the disjoint union $\bigsqcup_{x \in X} \partial\mathfrak{G}_x$ is a groupoid (i.e., is closed under taking compositions and inverses), and depends only on $\mathfrak{G}$ and X. Restriction of $\mathfrak{d}\mathfrak{G}_X$ onto any set $\bigsqcup_{x \in Y} \partial\mathfrak{G}_x$ for $Y \subset X$ does not depend on X.*

Proof. If x, y are units of $\mathfrak{G}$ such that R_x^y is defined, then restriction of the transformation R_x^y onto $\partial\mathfrak{G}_x$ is equal to the transformation $\xi \mapsto [\xi, y]_V$, where V is a neighborhood of points x, y of diameter less than δ_0, and $[\cdot,\cdot]_V$ is the function given in (3).

The groupoid $\mathfrak{H}$ of germs of the pseudogroup acting on the disjoint union $\bigsqcup_{x\in X} \partial\mathfrak{G}_x$ and generated by the transformations $[\cdot,\cdot]_V$ and $\xi \mapsto \xi \cdot g$ is equivalent to the projection $\mathsf{P}_-(\partial\mathfrak{G} \rtimes \mathfrak{G})$ of the geodesic quasi-flow onto the direction of the boundaries of the Cayley graphs. Every element of the groupoid $\mathfrak{H}$ is, by [21, Lemma 6.7], equal to a composition of $\xi \mapsto \xi \cdot g$, followed by $\xi \mapsto [x, \xi]_U$, and then by $\xi \mapsto \xi \cdot h$ for some $g, h \in \mathfrak{G}$.

Restriction of the transformation R_g^h onto an open subset of $\partial\mathfrak{G}_{\mathsf{o}(g)}$ is equal to composition of the transformations

$$\xi \mapsto \xi \cdot g^{-1} \mapsto [\mathsf{t}(h), \xi \cdot g^{-1}]_V \mapsto [\mathsf{t}(h), \xi \cdot g^{-1}]_V \cdot h,$$

where V is a neighborhood of diameter less than δ_0 of $\mathsf{t}(g)$ in $\mathfrak{G}^{(0)}$.

It follows that the set of germs of R_g^h is equal to the groupoid $\mathfrak{H}$. The last statement of the theorem follows directly from the definitions. □

The dual groupoid $\mathfrak{G}^\top$ of a hyperbolic groupoid $\mathfrak{G}$ is defined in [21] as the projection $\mathsf{P}_-(\partial\mathfrak{G} \rtimes \mathfrak{G})$, see [21, Definition 6.4]. It follows from the proof of Theorem 3.3 that this definition is equivalent to the following.

Definition 3.1. Let $\mathfrak{G}$ be a hyperbolic groupoid. The *dual groupoid* $\mathfrak{G}^\top$ is any groupoid equivalent to $\mathfrak{d}\mathfrak{G}_X$.

The groupoid $\mathfrak{d}\mathfrak{G}_X$ is not second countable, but it is equivalent to a second countable groupoid.

4. Minimal hyperbolic groupoids

Let $\mathfrak{G}$ be a hyperbolic groupoid and let (S, X) be its generating pair. We say that a Cayley graph $\mathfrak{G}(x, S)$ is *topologically mixing* if for every point $\xi \in \partial\mathfrak{G}_x$ and every neighborhood U of ξ in $\overline{\mathfrak{G}_x^X}$ the set of accumulation points of $\mathsf{t}(U \cap \mathfrak{G}_x^X)$ contains the interior of X.

Proposition 4.1. *Let $\mathfrak{G}$ be a hyperbolic groupoid. Then the following conditions are equivalent.*

1. *Some Cayley graph of $\mathfrak{G}$ is topologically mixing.*
2. *Every Cayley graph of $\mathfrak{G}$ is topologically mixing.*
3. *Every $\mathfrak{G}$-orbit is dense in $\mathfrak{G}^{(0)}$.*

Proof. Note that (2) obviously implies (3). It remains to prove that (1) implies (2) and that (3) implies (1).

Let us show that (1) implies (2). We will split the proof into four lemmas.

Lemma 4.2. *If $\mathfrak{G}(x, S)$ is topologically mixing, then $\mathfrak{G}(x, S')$ is topologically mixing for any generating set S' of $\mathfrak{G}|_X$.*

Proof. The Lipschitz class of the log-scale on $\overline{\mathfrak{G}_x^X}$ does not depend on the choice of the generating set, hence the set of open neighborhoods of points of $\overline{\mathfrak{G}_x^X}$ does not depend on S. □

We may assume now that all generating pairs in our proof satisfy the conditions of Proposition 2.2.

Lemma 4.3. *Let (X', S') be a compact generating pair of $\mathfrak{G}$ such that $\mathfrak{G}(x, S')$ is topologically mixing. Let (X, S) be a generating pair such that $X \subset X'$. Then $\mathfrak{G}(x, S)$ is topologically mixing.*

Proof. For every neighborhood W of a point $\xi \in \partial\mathfrak{G}_x$ in the Cayley graph $\mathfrak{G}(x, S')$ and every point y of the interior of X, the point y is a limit of a sequence of $\mathsf{t}(g_n)$ for pairwise different $g_n \in W$. Since y is an internal point of X, for all sufficiently big n we will have $\mathsf{t}(g_n) \in X$, hence $g_n \in \mathfrak{G}|_X$. Consequently, the set of accumulation points of $\mathsf{t}(W \cap \mathfrak{G}|_X)$ contains the interior of X, which shows that $\mathfrak{G}(x, S)$ is topologically mixing. □

Lemma 4.4. *Let (X, S) and (X', S') be compact generating pairs of $\mathfrak{G}$ such that $X \subset X'$ and $\mathfrak{G}(x, S)$ is topologically mixing. Then $\mathfrak{G}(x, S')$ is topologically mixing.*

Proof. There exists a finite collection $\mathcal{U}$ of elements $U \in \widetilde{\mathfrak{G}}$ such that $\mathsf{o}(U)$ cover X' and $\mathsf{t}(U) \subset X$. Denote $A = \bigcup_{U \in \mathcal{U}} U \cap \mathfrak{G}|_{X'}$. It follows from elementary properties of Gromov hyperbolic graphs that for every neighborhood W' of $\xi \in \partial\mathfrak{G}_x$ in $\overline{\mathfrak{G}_x^{X'}}$ there exists a neighborhood W of ξ in $\overline{\mathfrak{G}_x^{X}}$ such that for every $g \in W$ and $h \in A$ we have $h^{-1}g \in W'$. Let y be an arbitrary internal point of X'. Let $U \in \mathcal{U}$ be such that $y \in \mathsf{o}(U)$. Then $U(y)$ is an internal point of X, and hence it is a limit of a sequence of points of the form $\mathsf{t}(g_n)$ for a sequence of different elements $g_n \in W$. Then $\mathsf{t}(g_n)$ belongs to $\mathsf{t}(U)$ and $U^{-1}(\mathsf{t}(g_n))$ belongs to X' for all sufficiently big n. If h is the germ of U at $U^{-1}(\mathsf{t}(g_n))$, then $h \in A$, $g_n h^{-1}$ belongs to W', and y is equal to the limit of the sequence $\mathsf{t}(h^{-1}g_n)$. Consequently, the Cayley graph $\mathfrak{G}(x, S')$ is also topologically mixing. □

The following lemma will finish the proof that (1) implies (2).

Lemma 4.5. *If a Cayley graph $\mathfrak{G}(x, S)$ is topologically mixing for some $x \in X$, then the Cayley graph $\mathfrak{G}(x', S)$ is topologically mixing for every $x' \in X$.*

Proof. Since the property of being topologically mixing does not depend on the choice of the generating set S, we may assume that S satisfies the conditions of Proposition 2.2. Let $\mathcal{S}$ be a finite open covering of S by positive contracting elements of $\widetilde{\mathfrak{G}}$.

Note also that if $\mathfrak{G}(x, S)$ is topologically mixing, then for any point $y \in X$ in the orbit of x the Cayley graph $\mathfrak{G}(y, S)$ is topologically mixing. It follows that we may assume that x' belongs to an open transversal $X_0 \subset X$ and that x is arbitrarily close to x' (since the orbit of x is dense in the interior of X).

Let $\xi' \in \partial\mathfrak{G}_{x'}$ be an arbitrary point. Let $X' \supset X$ be a compact set such that $\mathsf{t}(F)$ is contained in X' for every $F \in \mathcal{S}$. Let W be a neighborhood of ξ' in $\overline{\mathfrak{G}_{x'}^{X'}}$. Note that then the set of values of $R_x^{x'}$ belongs to $\overline{\mathfrak{G}_{x'}^{X'}}$. Denote $\xi = R_{x'}^{x}(\xi')$. We have $R_x^{x'}(\xi) = \xi'$. There exists $g \in \mathfrak{G}_x$ such that $\overline{T_g}$ is a neighborhood of ξ in $\partial\mathfrak{G}_x$ and $R_x^{x'}(\mathcal{T}_g) \subset W \cap \partial\mathfrak{G}_{x'}$. The set $\overline{T_g}$ is a neighborhood of ξ in $\overline{\mathfrak{G}_x^{X}}$, hence $\mathsf{t}(T_g)$ is dense in the interior of X. Denote $T_g' = R_x^{x'}(T_g)$. The set of accumulation points of T_g in $\overline{\mathfrak{G}_x^{X}}$ is equal to $\mathcal{T}_g$. The set of accumulation points of T_g' in $\overline{\mathfrak{G}_x^{X'}}$ is $R_x^{x'}(\mathcal{T}_g)$. Consequently, all but finitely many points of T_g' belong to W. But the set of accumulation points of $\mathsf{t}(T_g)$ is equal to the set of accumulation points of $\mathsf{t}(T_g')$, due to the fact that the elements of $\mathcal{S}$ are contracting. Consequently, the set of accumulation points of $\mathsf{t}(W \cap \mathfrak{G}_{x'})$ contains the set of accumulation points of $\mathsf{t}(T_g)$,

hence it contains the interior of X. We have shown that for every neighborhood W of ξ' in $\overline{\mathfrak{G}_{x'}^{X'}}$ the set of accumulation points of $\mathsf{t}(W \cap \mathfrak{G}_{x'})$ contains the interior of X. Repeating the argument from the proof of Lemma 4.4, we conclude that the set of accumulation points of $\mathsf{t}(W \cap \mathfrak{G}_{x'})$ contains the interior of X'. □

Finally, let us show now that (3) implies (1). Let $\mathfrak{G}(x, S)$ be a Cayley graph, where (S, X) is a generating pair. We have to show that $\mathsf{t}(\mathfrak{G}_x^X)$ is dense in the interior of X. Let $U \subset X$ be an arbitrary open subset. Since every orbit of $\mathfrak{G}$ is dense, U is a transversal. Consequently, the set of elements $g \in \mathfrak{G}_x^X$ such that $\mathsf{t}(g) \in U$ is a net in the Cayley graph $\mathfrak{G}(x, S)$. This implies that for every neighborhood W in $\overline{\mathfrak{G}_x^X}$ the set $\mathsf{t}(W)$ intersects U. □

Definition 4.1. We say that a hyperbolic groupoid $\mathfrak{G}$ is *minimal* if it satisfies the equivalent conditions of Proposition 4.1.

Proposition 4.6. *If $\mathfrak{G}$ is minimal, then the groupoid $\mathfrak{d}\mathfrak{G}_x$ equal to restriction of $\mathfrak{d}\mathfrak{G}_X$ onto $\partial\mathfrak{G}_x$ is equivalent to the groupoid $\mathfrak{d}\mathfrak{G}_X$.*

Recall that $\mathfrak{d}\mathfrak{G}_x$ depends only on x and $\mathfrak{G}$.

Proof. The set $\partial\mathfrak{G}_x$ is an open subset of the space of units of the groupoid $\mathfrak{d}\mathfrak{G}_X$. It follows from minimality that for every $y \in \mathfrak{G}^{(0)}$ there exist elements $g \in \mathfrak{G}$ and $x \in X$ such that $\mathsf{o}(g) = y$ and $|\mathsf{t}(g) - x| < \delta_0$. Consequently, $\partial\mathfrak{G}_x$ is a $\mathfrak{d}\mathfrak{G}_X$-transversal. □

Hence, we can define the dual groupoid $\mathfrak{G}^\top$ as the groupoid of germs of restrictions of the maps R_g^h onto open subsets of $\partial\mathfrak{G}_x$.

Proposition 4.7. *If $\mathfrak{G}$ is minimal, then $\mathfrak{G}^\top$ is also minimal.*

Proof. It is enough to show that the groupoid $\mathfrak{d}\mathfrak{G}_x$ is minimal. By definition of topologically mixing Cayley graphs, for any $g, h \in \mathfrak{G}_x^{X_0}$ and any neighborhood U of $\mathsf{t}(g)$ there exists $f \in T_h$ such that $\mathsf{t}(f) \in U$ and $\mathcal{T}_f \subset \mathcal{T}_h$. Then restriction of $R_g^f : \mathcal{T}_g \longrightarrow \mathcal{T}_h$ onto the interior of $\mathcal{T}_g$ is an element of $\widetilde{\mathfrak{d}\mathfrak{G}_X}$. It follows that orbit of any internal point of $\mathcal{T}_g$ is dense. □

For definition of a locally diagonal groupoid, see [21, Definition 3.13]. We will need the next proposition only as a technical condition for the Theorem 4.9.

Proposition 4.8. *If $\mathfrak{G}$ is minimal, then the geodesic quasi-flow $\partial\mathfrak{G} \rtimes \mathfrak{G}$ is locally diagonal.*

Proof. Consider the covering of $\partial\mathfrak{G} \rtimes \mathfrak{G}$ by the rectangles $R_{U,g}$, as defined in the proof of [21, Theorem 5.1]. Suppose that this covering does not satisfy the conditions of [21, Definition 3.13]. Then there exist a rectangle $R_{U,g}$ and a non-unit element $h \in \mathfrak{G}$ such that $[\mathsf{o}(h), \xi]_U = \xi \cdot h$ for all $\xi \in \mathcal{T}_g^\circ$.

Let $\xi = \ldots g_2 g_1 \cdot g$ for $g_i \in S$ be an arbitrary point of $\mathcal{T}_g^\circ$, and let $G_i \in \mathcal{S}$ be such that g_i is ϵ-contained in G_i. Then $\mathsf{o}(h), \mathsf{t}(h) \in \mathsf{o}(U)$, and $[\mathsf{o}(h), \xi]_U$ is represented by the product $\ldots G_2 G_1 \cdot (U, \mathsf{o}(h))$. If $g' = (U, \mathsf{o}(h))$ and g_i' are the germs of G_i such that $\ldots G_2 G_1 \cdot (U, \mathsf{o}(h)) = \ldots g_2' g_1' \cdot g'$, then $\ldots g_2 g_1 \cdot gh = \ldots g_2' g_1' \cdot g'$.

There exists a constant Δ such that for any element $h \in \mathfrak{G}$ which has a trivial projection on the second direction of the geodesic quasi-flow we have $|\nu(h)| \leq \Delta$. This follows, for example, from Proposition 3.2. Let Q be the set of elements (ξ, h)

of the geodesic quasi-flow such that $\xi \in \mathcal{T}_{\mathsf{t}(h)}$, $\xi \cdot h \in \mathcal{T}_{\mathsf{o}(h)}$, and $|\nu(h)| \leq \Delta$. Then Q has compact closure.

Consider the sequence

$$h_0 = gh(g')^{-1} = UhU^{-1},$$
$$h_1 = g_1gh(g_1'g')^{-1} = G_1Uh(G_1U)^{-1},$$
$$\vdots$$
$$h_n = g_n \cdots g_1gh(g_n' \cdots g_1'g')^{-1} = G_n \cdots G_1Uh(G_n \cdots G_1U)^{-1}.$$

For each element h_n we have

$$\ldots g_{n+2}g_{n+1} \cdot h_n = \ldots g_{n+2}'g_{n+2}'$$

and

$$[\mathsf{o}(h_n), \ldots g_{n+2}g_{n+1}]_{G_n \cdots G_1U} = \ldots g_{n+2}g_{n+1}.$$

There exists n_0 such that $\mathcal{T}_{g_n \cdots g_1 \cdot g} \subset \mathcal{T}_g^\circ$ for all $n \geq n_0$. It follows that all elements h_n have trivial projections onto the second coordinate of the geodesic quasi-flow. Note that all elements h_n are non-units and belong to Q for $n \geq 1$. Let H be the union of the sequences h_n for all possible choices of $\xi \in \mathcal{T}_g^\circ$, representations $\xi = \ldots g_2g_1 \cdot g$, and $G_i \in \mathcal{S}$.

Then by definition of minimality, the sets $\mathsf{o}(H)$ and $\mathsf{t}(H)$ are dense in the interior of X. Note also that $|\mathsf{o}(h_n) - \mathsf{t}(h_n)| \to 0$ as $n \to \infty$.

The set Q can be covered by a finite set $\mathcal{U}$ of extendable local homeomorphisms $U \in \widetilde{\mathfrak{G}}$. Choose for each $U \in \mathcal{U}$ an extension $\widehat{U} \in \widetilde{\mathfrak{G}}$ such that $\overline{U} \subset \widehat{U}$. For every $U \in \mathcal{U}$ the set of points $x \in \mathsf{o}(\overline{U})$ such that the germ $(\widehat{U}, x)$ is non-trivial and $\overline{U}(x) = x$, is nowhere dense. It follows that there exists an internal point x of X such that for every $U \in \mathcal{U}$ either (U, x) is trivial (hence x belongs to an open subset of the set of fixed points of U), or $U(x) \neq x$. Then there exist numbers $r > 0$ and $\epsilon_0 > 0$ such that for every $g \in Q$ such that $|x - \mathsf{o}(g)| < r$ either g is trivial, or $|\mathsf{o}(g) - \mathsf{t}(g)| > \epsilon_0$.

But then we get a contradiction, since there will exist $h_n \in H$ such that $|x - \mathsf{o}(h_n)| < r$ and $|\mathsf{o}(h_n) - \mathsf{t}(h_n)| < \epsilon_0$. □

The following theorem is then a direct corollary of Proposition 4.8 and [21, Theorem 6.15].

Theorem 4.9. *Let $\mathfrak{G}$ be a minimal hyperbolic groupoid. Then the groupoid $\mathfrak{G}^\top$ is hyperbolic, and the groupoid $(\mathfrak{G}^\top)^\top$ is equivalent to $\mathfrak{G}$.*

5. Hyperbolic metric

It is proved in [21, Proposition 4.11] that the coarse equivalence class of a Busemann cocycle on a hyperbolic groupoid $\mathfrak{G}$ is uniquely determined by the topological groupoid $\mathfrak{G}$. A hyperbolic groupoids together with a *strong* equivalence class of the cocycle is a *graded hyperbolic groupoid*. In different situations different gradings are natural.

Example 5.1. Let f be a hyperbolic complex rational function. Then f is expanding with respect to a Riemanian metric on a neighborhood of the Julia set of f, hence the groupoid $\mathfrak{F}$ generated by the germs of the action of f on its Julia set is hyperbolic, where the grading $\nu : \mathfrak{F} \longrightarrow \mathbb{Z}$ is given by the degree of germs.

Namely, every element of $\mathfrak{F}$ is a composition $g = (f^n, x)^{-1} \circ (f^m, y)$ for some points x, y of the Julia set. We define then $\nu(g) = n - m$. The Cayley graphs of $\mathfrak{F}$ are regular trees and ν is equal to the Busemann cocycle associated with the point of its boundary given by the path $x, f(x), f^2(x), \ldots$.

On the other hand, it is easy to see that the map

$$\nu_1((F, x)) = -\ln |F'(x)|, \tag{4}$$

for $F \in \widetilde{\mathfrak{F}}$, is a cocycle coarsely equivalent to ν. Consequently, $(\mathfrak{F}, \nu_1)$ is also a graded hyperbolic groupoid.

If $(\mathfrak{G}, \nu)$ is a graded hyperbolic groupoid, then $(\partial\mathfrak{G} \rtimes \mathfrak{G}, \widetilde{\nu})$ is a graded groupoid where $\widetilde{\nu}(g, \xi) = \nu(g)$ is the lift of the quasi-cocycle ν to the geodesic quasi-flow. The graded groupoid $(\partial\mathfrak{G} \rtimes \mathfrak{G}, \widetilde{\nu})$ is uniquely determined by the graded groupoid $(\mathfrak{G}, \nu)$.

On the other hand, by [21, Theorem 6.6], if $(\mathfrak{H}, \nu)$ is a Smale quasi-flow, then its projections $\mathsf{P}_+(\mathfrak{H})$ and $\mathsf{P}_-(\mathfrak{H})$ onto the stable and unstable directions are hyperbolic, and there exist quasi-cocycles ν_+ and ν_- on $\mathsf{P}_+(\mathfrak{H})$ and $\mathsf{P}_-(\mathfrak{H})$ such that the functions $|\nu_+(\mathsf{P}_+(g)) - \nu(g)|$ and $|\nu_-(\mathsf{P}_-(g)) + \nu(g)|$ are uniformly bounded.

We get hence the following summary of the above facts.

Proposition 5.1. *Let $(\mathfrak{G}, \nu)$ be a minimal graded hyperbolic groupoid. There exist unique, up to strong equivalence, quasi-cocycles $\widetilde{\nu}$ and $\nu^\top$ on $\partial\mathfrak{G} \rtimes \mathfrak{G}$ and $\mathfrak{G}^\top$ such that $|\nu(\mathsf{P}_+(g)) - \widetilde{\nu}(g)|$ and $|\nu^\top(\mathsf{P}_-(g)) + \widetilde{\nu}(g)|$ are uniformly bounded.*

It follows directly from the definitions that for every germ (R_g^h, ξ) we have

$$\nu^\top(R_h^g, \xi) \doteq |\nu(g) - \nu(h)|. \tag{5}$$

Definition 5.1. Let $(\mathfrak{G}, \nu)$ be a graded hyperbolic groupoid with locally diagonal geodesic quasi-flow. Then the *dual graded groupoid* $(\mathfrak{G}, \nu)^\top$ is the groupoid $(\mathfrak{G}^\top, \nu^\top)$ where $\mathfrak{G}^\top$ is the hyperbolic groupoid dual to $\mathfrak{G}$, and the quasi-cocycle $\nu^\top : \mathfrak{G}^\top \longrightarrow \mathbb{R}$ is equal to the projection of the quasi-cocycle $-\widetilde{\nu} : \partial\mathfrak{G} \rtimes \mathfrak{G} \longrightarrow \mathbb{R}$, where $\widetilde{\nu}$ is the lift of ν.

Let now $(\mathfrak{G}, \nu)$ be a minimal graded hyperbolic groupoid. The space of units of $\mathfrak{G}^\top$ is locally homeomorphic to boundaries of the Cayley graphs $\mathfrak{G}(x, S)$ of $\mathfrak{G}$. The quasi-cocycle ν defines a natural log-scale on $\partial\mathfrak{G}_x$ by rule that $\ell_\nu(\xi_1, \xi_2)$ is equal to the minimal value of ν along a geodesic path in $\mathfrak{G}(x, S)$ connecting ξ_1 to ξ_2 (see [21, Subsection 4.4]).

This log-scale satisfies an estimate (see Proposition 3.2)

$$\ell_\nu(R_h^g(\xi_1), R_h^g(\xi_2)) \doteq \ell_\nu(\xi_1, \xi_2) + \nu(g) - \nu(h) \tag{6}$$

for all $\xi_1, \xi_2 \in \overline{T_h}$.

It follows that we get a Lipschitz structure on $\mathfrak{G}^\top$ that is uniquely determined (up to bi-Lipschitz equivalence) by the grading ν. We will call it the *hyperbolic log-scale associated with* ν. Since ν and $\nu^\top$ uniquely determine each other, we will also say that the defined log-scale is associated with $\nu^\top$ (if there is no confusion on which of the groupoids $\mathfrak{G}$ and $\mathfrak{G}^\top$ the log-scale is defined).

Estimates (6) and (5) immediately imply the following proposition.

Proposition 5.2. *Let ℓ be the hyperbolic log-scale on $\mathfrak{G}$ associated with the grading ν of the hyperbolic groupoid $(\mathfrak{G}, \nu)$. Then there exists a constant c such that for every*

$g \in \mathfrak{G}$ there exists a neighborhood $U \in \widetilde{\mathfrak{G}}$ such that for any two points $x, y \in \mathsf{o}(U)$ we have

$$|\ell(U(x), U(y)) - (\ell(x, y) + \nu(g))| < c.$$

Recall that we say that a metric $|\cdot|$ is *associated* with a log-scale ℓ if there exists a constant $\alpha > 0$ such that

$$|x - y| \asymp e^{-\alpha \cdot \ell(x,y)}$$

for all pairs of points x, y. We call α the *exponent* of the associated metric. For every log-scale there exists a constant α_0 such that an associated metric exists for every positive exponent $\alpha < \alpha_0$.

Definition 5.2. A *(hyperbolic) metric of exponent* α associated with the quasi-cocycle ν is a metric of exponent α associated with the hyperbolic log-scale ℓ_ν.

The hyperbolic metric is called sometimes *visual*. Note that, by definition, a metric locally bi-Lipschitz equivalent to a hyperbolic metric of exponent α is also a hyperbolic metric of exponent α.

Thus, a grading ν of a hyperbolic groupoid determines for every positive sufficiently close to zero number α a unique locally bi-Lipschitz class of hyperbolic metrics. Proposition 5.2 is reformulated then as follows.

Proposition 5.3. *Let $|\cdot|$ be a hyperbolic metric of exponent α on a graded hyperbolic groupoid $(\mathfrak{G}, \nu)$. Then there exists a constant $c > 1$ such that for every $g \in \mathfrak{G}$ there exists a neighborhood $U \in \widetilde{\mathfrak{G}}$ such that for every pair of different points $x, y \in \mathsf{o}(U)$ we have*

$$c^{-1} e^{-\alpha \cdot \nu(g)} \le \frac{|U(x) - U(y)|}{|x - y|} \le c e^{-\alpha \cdot \nu(g)}.$$

In other words, the quasi-cocycle $\nu(g)$ is proportional, up to an additive constant, to the logarithm of the scaling factor of the germ g.

We will need a more precise version of the last proposition.

Proposition 5.4. *Let $|\cdot|$ be a hyperbolic metric of exponent α. Let S be a compact positive subset of $\mathfrak{G}$. Let $\mathcal{S}$ be a finite covering of S by λ-contracting positive elements of $\widetilde{\mathfrak{G}}$, where $\lambda \in (0, 1)$ is a fixed constant.*

Then there exist constants $\epsilon > 0$ and $c > 1$ such that if $g_1 g_2 \cdots g_n$ is a product of elements of S, and $F_i \in \mathcal{S}$ are such that g_i is ϵ-contained in F_i, then for any two points $x, y \in \mathsf{o}(F_1 \cdots F_n)$ on distance less than ϵ from $\mathsf{o}(g_n)$ we have

$$c^{-1} e^{-\alpha\nu(g_1 \cdots g_n)} |x - y| \le |F_1 \cdots F_n(x) - F_1 \cdots F_n(y)| \le c e^{-\alpha\nu(g_1 \cdots g_n)} |x - y|. \tag{7}$$

Proof. Note that if $\mathcal{S}'$ is a finite covering of S subordinate to a covering $\mathcal{S}$, then the statement of the proposition holds for $\mathcal{S}$ if and only if it holds for $\mathcal{S}'$. Since any two open coverings of S have a common finite subordinate covering, if the statement is true for some covering $\mathcal{S}$, then it is true for all coverings.

Let (S_1, X_1) be a generating pair of $\mathfrak{G}$ satisfying Proposition 2.2 and let $S \subset \mathfrak{G}$ be any positive contracting set. Assume that S_1 satisfies the conditions of the proposition. Let us show that it is satisfied for S.

Let (S_2, X_2) be a generating pair of $\mathfrak{G}$ such that $S \subset \mathfrak{G}|_{X_2}$ and $X_1 \subset X_2$. Then $\mathfrak{G}_x^{X_1}$ is a net in $\mathfrak{G}_x^{X_2}$. Consequently, every element of $\mathfrak{G}_{X_2}$ can be represented as a product $a_1 g a_2$, where $g \in \mathfrak{G}_{X_1}$ and a_1, a_2 belong to a fixed compact set $Q_1 \subset \mathfrak{G}$.

Every path corresponding to a finite or infinite product $\cdots g_2 g_1$ of elements of S is a quasi-geodesic (in a uniform way in the corresponding Cayley graph $\mathfrak{G}(\mathsf{o}(g_1), S_2)$).

If the path is infinite, then it converges to a point of $\partial\mathfrak{G}_{\mathsf{o}(g_1)}$. Since S_1 satisfies the conditions of Proposition 2.2, there exists a compact set $Q \subset \mathfrak{G}$ such that every product $g_n \cdots g_1 \in S^n$ can be represented in the form $a_1 \cdot s_m \cdots s_1 \cdot a_2$ for $a_1, a_2 \in Q$ and $s_i \in S_1$.

Let $\mathcal{Q}$ be a finite covering of Q by bi-Lipschitz elements of $\widetilde{\mathfrak{G}}$, and let $\mathcal{S} \subset \widetilde{\mathfrak{G}}$ be a finite covering of $S \cup S_1$ by contractions. Then there exists $\epsilon > 0$ such that if we have $g_n \cdots g_1 = h_0 s_m \cdots s_1 h_1$ for $g_i \in S$, $s_i \in S_1$, $h_i \in Q$, and if $G_i \in \mathcal{S}$, $H_i \in \mathcal{Q}$, and $U_i \in \mathcal{S}$ are such that g_i, h_i, and s_i are ϵ-contained in G_i, H_i, and U_i, respectively, then the compositions $G_n \cdots G_1$ and $H_0 U_m \cdots U_1 H_1$ coincide on the ϵ-neighborhood of $\mathsf{o}(g_1) = \mathsf{o}(h_1)$. It follows that the proposition holds for S.

Consequently, it is enough to prove the proposition for any groupoid equivalent to $\mathfrak{G}$.

We can represent $\mathfrak{G}$ as projection $\mathsf{P}_+(\partial\mathfrak{G} \rtimes \mathfrak{G})$ of the geodesic quasi-flow and use a generating set equal to projection of the generating set satisfying conditions of Proposition 2.4. Then the statement of the proposition will follow from (6) and the fact that the quasi-cocycle $\widetilde{\nu} : \partial\mathfrak{G} \rtimes \mathfrak{G} \longrightarrow \mathbb{R}$ agrees with the local products structure. □

Corollary 5.5. *Let S be a positive compact subset of a hyperbolic groupoid $(\mathfrak{G}, \nu)$, let $\mathcal{S}$ be a finite covering of S by positive contracting elements of $\widetilde{\mathfrak{G}}$. Then there exist $c > 0$ and $\epsilon > 0$ such that for any product $U = F_1 \cdots F_n$ of elements of $\mathcal{S}$ and for any two germs g_1, g_2 of U such that $|\mathsf{o}(g_1) - \mathsf{o}(g_2)| < \epsilon$ we have*

$$|\nu(g_1) - \nu(g_2)| < c.$$

Proof. There exist $\epsilon > 0$ and c such that for every product $U = F_1 \cdots F_n$ of elements of $\mathcal{S}$ and every pair $x_1, x_2 \in \mathsf{o}(U)$ for $F_i \in \mathcal{S}$, such that $|x_1 - x_2| < \epsilon$ we have $c^{-1} e^{-\alpha\nu(U, x_i)} |x_1 - x_2| < |U(x_1) - U(x_2)| < c e^{-\alpha\nu(U, x_i)} |x_1 - x_2|$ for $i = 1, 2$. It implies $|\nu(U, x_1) - \nu(U, x_2)| < 2\alpha^{-1} \ln c$. □

Theorem 5.6. *Let $(\mathfrak{G}, \nu)$ be a graded hyperbolic groupoid, let (S, X) be a generating pair satisfying the conditions of Proposition 2.2 (for an arbitrary metric $|\cdot|$). A metric $|\cdot|_1$ defined on a neighborhood of X is a hyperbolic metric of exponent α if and only if there exists a finite covering $\mathcal{S}$ of S by positive elements of $\widetilde{\mathfrak{G}}$ such that for every product $F_1 \cdots F_n$ of elements of $\mathcal{S}$ we have*

$$|F_1 \cdots F_n(x) - F_1 \cdots F_n(y)|_1 \asymp e^{-\alpha\nu(g)} |x - y|_1,$$

for all $x, y \in \mathsf{o}(F_1 \cdots F_n)$, where g is any germ of $F_1 \cdots F_n$ and the coefficients in the estimate do not depend on n, F_i, g, x, and y.

Proof. Proposition 5.3 implies the 'if' part of the theorem. In order to prove the theorem in the other direction, is enough to show that if $|\cdot|_1$ and $|\cdot|_2$ are metrics satisfying the conditions of the theorem, then they are locally bi-Lipschitz equivalent.

There exists a covering $\mathcal{S}$ of S satisfying the condition of the theorem for both metric $|\cdot|_i$. Let ϵ be a common Lebesgue number of the covering $\mathcal{S}$ for both metrics.

Let $x \in X$ be an arbitrary point. Let Δ be an upper bound on the value of the cocycle ν on elements of S . For every n there exists a sequence $g_1, \ldots, g_k$ of elements of S such that $\mathsf{t}(g_1 \cdots g_k) = x$, and $n \le \nu(g_1 \cdots g_k) < n + \Delta + \eta$. Let $G_i \in \mathcal{S}$ be such that g_i is ϵ-contained in G_i. Then, for any $i = 1, 2$, if

$$|x - y|_i < c^{-1} e^{-\alpha(n + \Delta + \eta)} \epsilon = c^{-1} e^{-\alpha(\Delta + \eta)} \epsilon \cdot e^{-\alpha n},$$

then $(G_1 \cdots G_k)^{-1}(y)$ is defined. Here $c > 1$ is a sufficiently big constant.

On the other hand, if $(G_1 \cdots G_k)^{-1}(y)$ is defined, then $|x-y|_i < cDe^{-\alpha n}$, where D is an upper bound on the diameters of the sets $\mathsf{o}(G_i)$.

Since $|x-y|_1 < c^{-1}e^{-\alpha(\Delta+\eta)}\epsilon \cdot e^{-\alpha n}$ for $n = \left\lfloor \frac{\ln\left(ce^{\alpha(\Delta+\eta)}\epsilon^{-1}\cdot|x-y|_1\right)}{-\alpha} \right\rfloor$, we have for all x, y, such that $|x-y|$ is small enough

$$|x-y|_2 < cD\exp\left(-\alpha\left\lfloor \frac{\ln\left(ce^{\alpha(\Delta+\eta)}\epsilon^{-1}\cdot|x-y|_1\right)}{-\alpha} \right\rfloor\right) <$$
$$cD\exp\left(\ln\left(ce^{\alpha(\Delta+\eta)}\epsilon^{-1}\cdot|x-y|_1\right)+\alpha\right) = c^2D\epsilon^{-1}e^{\alpha(\Delta+\eta+1)}\cdot|x-y|_1,$$

hence $|\cdot|_1$ and $|\cdot|_2$ are locally bi-Lipschitz equivalent. □

As an example of application of the previous theorem, consider the following.

Proposition 5.7. *Let $\widetilde{\mathfrak{G}}$ a pseudogroup acting on a subset X of $\mathbb{C}$ by biholomorphic maps. Define $\nu(F, z) = -\ln|F'(z)|$ and suppose that $(\mathfrak{G}, \nu)$ is hyperbolic. Then the usual metric $|z_1 - z_2|$ on $\mathbb{C}$ is a hyperbolic metric on X of exponent* 1.

In particular, if $\widetilde{\mathfrak{G}}$ is the groupoid generated by the restriction of a hyperbolic complex rational function onto its Julia set, then the usual metric on $\mathbb{C}$ (restricted to the Julia set) is a hyperbolic metric for the groupoid $(\mathfrak{G}, \nu)$, where ν is as in Proposition 5.7.

Proof. Let S be a compact subset of $\mathfrak{G}$ such that $\nu(s) > 0$ for all $s \in S$. Then $\nu(s)$ for $s \in S$ is bounded from below by a positive constant. Let $\mathcal{S}$ be a finite open covering of S by relatively compact extendable λ-contracting elements of $\widetilde{\mathfrak{G}}$, where $0 < \lambda < 1$.

Since the set $\mathcal{S}$ is finite, and $F \in \mathcal{S}$ are relatively compact and extendable, there exist constants $\epsilon > 0$ and $c_1 > 0$ such that for every $F \in \mathcal{S}$ and any $x, y \in \mathsf{o}(F)$ such that $|x-y| < \epsilon$ we have

$$\left|\frac{F(x)-F(y)}{(x-y)F'(x)} - 1\right| < c_1|x-y|.$$

Taking ϵ small enough, we may assume that $c_1\epsilon < 1$, then

$$0 < 1 - c_1|x-y| < \left|\frac{F(x)-F(y)}{(x-y)F'(x)}\right| < 1 + c_1|x-y|$$

We also assume that ϵ is less than the Lebesgue number of the covering $\mathcal{S}$.

Let us show that ϵ satisfies then the conditions of Theorem 5.6. Suppose that $g_n g_{n-1} \cdots g_1$ is a product of elements of S, and $F_i \in \mathcal{S}$ are such that g_i is ϵ-contained in F_i. Let x_0, y_0 be points on distance less than ϵ from $\mathsf{o}(g_1)$. Denote $x_k = F_k \cdots F_1(x_0)$ and $y_k = F_k \cdots F_1(y_0)$. Then $|x_k - y_k| < \epsilon\lambda^k$.

We have

$$|(F_n \cdots F_1)'(x_0)| = |F_n'(x_{n-1})| \cdot |F_{n-1}'(x_{n-2})| \cdots |F_1'(x_0)| =$$
$$\left|\frac{F_n'(x_{n-1})(x_{n-1}-y_{n-1})}{x_n - y_n}\right| \cdot \left|\frac{F_{n-1}'(x_{n-2})(x_{n-2}-y_{n-2})}{x_{n-1}-y_{n-1}}\right| \cdots \left|\frac{F_1'(x_0)(x_0-y_0)}{x_1-y_1}\right| \cdot \left|\frac{x_n-y_n}{x_0-y_0}\right|.$$

The last product is less than

$$(1-c_1|x_{n-1}-y_{n-1}|)^{-1}(1-c_1|x_{n-2}-y_{n-2}|)^{-1}\cdots(1-c_1|x_0-y_0|)^{-1}\cdot\left|\frac{x_n-y_n}{x_0-y_0}\right| < \left|\frac{x_n-y_n}{x_0-y_0}\right|\prod_{k=0}^{\infty}(1-c_1\epsilon\lambda^k)^{-1},$$

where the infinite product is convergent. Similarly, $|(F_n\cdots F_1)'(x_0)|$ is bigger than

$$(1+c_1|x_{n-1}-y_{n-1}|)^{-1}(1+c_1|x_{n-2}-y_{n-2}|)^{-1}\cdots(1+c_1|x_0-y_0|)^{-1}\cdot\left|\frac{x_n-y_n}{x_0-y_0}\right| > \left|\frac{x_n-y_n}{x_0-y_0}\right|\prod_{k=0}^{\infty}(1+c_1\epsilon\lambda^k)^{-1}.$$

We have shown that there exists a constant $c_2>1$ such that for any two points x,y from the ϵ-neighborhood of $\mathsf{o}(g_1)$ we have

$$c_2^{-1}|U'(x)| < \left|\frac{U(x)-U(y)}{x-y}\right| < c_2|U'(x)|$$

where $U=F_n\cdots F_1$. It follows that $c_2^{-2}|U'(y)|<|U'(x)|<c_2^2|U'(y)|$ for all such x,y. In particular,

$$c_2^{-2}|U'(\mathsf{o}(g_1))| = c_2^{-2}e^{-\nu(g_n\cdots g_1)} < |U'(x)| < c_2^2|U'(\mathsf{o}(g_1))| = c_2^2e^{-\nu(g_n\cdots g_1)},$$

hence

$$c_2^{-3}e^{-\nu(g_n\cdots g_1)} < \left|\frac{U(x)-U(y)}{x-y}\right| < c_2^3e^{-\nu(g_n\cdots g_1)},$$

which implies by Theorem 5.6 that $|x-y|$ is a hyperbolic metric for the cocycle ν. ☐

6. Growth and Entropy

6.1. Growth of graded hyperbolic groupoids.

Definition 6.1. We say that a quasi-cocycle $\psi:\mathfrak{G}|_X\longrightarrow\mathbb{R}$ is *dualizable* if there exists a quasi-cocycle $-\psi^\top$ on $\mathfrak{G}^\top$ such that lifts of ψ and $-\psi^\top$ to the geodesic flow $\partial\mathfrak{G}\rtimes\mathfrak{G}$ are strongly equivalent.

Here lift of a quasi-cocycle ψ to the geodesic flow $\partial\mathfrak{G}\rtimes\mathfrak{G}$ is given by $\psi(\xi,g)=\psi(g)$. Recall that $\partial\mathfrak{G}\rtimes\mathfrak{G}$ is also equivalent to the geodesic flow of $\mathfrak{G}^\top$, by [21, Theorem 6.9].

Any Busemann quasi-cocycle is dualizable by [21, Theorem 6.6]. Another obvious example of a dualizable cocycle is the constant zero cocycle. We will see later that any *Hölder continuous* cocycle is dualizable.

Theorem 6.1. *Let $\mathfrak{G}$ be a minimal hyperbolic groupoid graded by a Busemann quasi-cocycle $\nu:\mathfrak{G}|_X\longrightarrow\mathbb{R}$. Let $\psi:\mathfrak{G}|_X\longrightarrow\mathbb{R}$ be a dualizable quasi-cocycle.*

There exists a positive number Δ and a number β such that for every $x\in X$ and for any compact neighborhood C of $\xi\in\partial\mathfrak{G}_x$ in $\overline{\mathfrak{G}_x^X}$ there exist positive constants k_1 and k_2 such that

$$k_1e^{\beta n}\le\sum_{g\in C\cap\mathfrak{G}_x,\,n-\Delta\le\nu(g)\le n}e^{\psi(g)}\le k_2e^{\beta n}$$

for all sufficiently large n.

Definition 6.2. The number β from Theorem 6.1 is called *pressure* of ψ relative to ν.

We are mostly interested in the case when ψ is constant zero, but the general case also has its applications.

Proof. Let us prove at first the following technical result.

Proposition 6.2. *Let $(\mathfrak{G}, \nu)$ be a minimal graded hyperbolic groupoid, and let X be a compact topological $\mathfrak{G}$-transversal. Then there exists a compact generating set S of $\mathfrak{G}|_X$ satisfying the conditions of Proposition 2.2, and such that there exists $r_0 > 0$ such that for every $x \in X$ the set $\mathcal{T}_x$ contains a point ζ_x such that the ball of $\overline{\mathfrak{G}_x^X}$ of radius r_0 (with respect to a metric associated with the natural log-scale on $\overline{\mathfrak{G}_x^X}$) and center in ζ_x is contained in $\overline{T_x}$.*

Proof. Let S_1 be a generating set satisfying conditions of Proposition 2.2. Denote by $\overline{T_g}^{(1)}$ and $\mathcal{T}_g^{(1)}$ the sets defined with respect to the set S_1. There exists a compact set $A \subset \mathfrak{G}|_X$ and a positive number r_1 such that for every $g \in \mathfrak{G}|_X$ the set $\bigcup_{a \in A \cap \mathfrak{G}_{\mathbf{t}g}} \overline{T_{ag}}^{(1)}$ contains the r_1-neighborhood of $\overline{T_g}^{(1)}$ (see Proposition 2.3). There exists then an integer $n > 0$ such that $S = S_1 \cup AS_1^n$ satisfies the conditions of Proposition 2.2. Then it will satisfy the condition of our proposition for any point $\zeta_x \in \mathcal{T}_x^{(1)} \subset \mathcal{T}_x$. □

Let $\eta > 0$ be such that ν and ψ are η-quasi-cocycles. Let (S, X) be a generating pair of $\mathfrak{G}$ satisfying Proposition 6.2. Let $\mathcal{S}$ be a finite covering of S by positive contracting elements of $\widetilde{\mathfrak{G}}$, and let δ_1 be a Lebesgue number of the covering $\mathcal{S}$.

Let D be an upper bound on values of ν on elements of S, and let $\Delta = D + \eta$. Recall that $\nu(g) > 2\eta$ for all $g \in S$. Then for every product $\ldots g_2 g_1$ of elements of S we have

$$\eta < \nu(g_n \cdots g_1) - \nu(g_{n-1} \cdots g_1) < \Delta.$$

Denote for $x \in X$ and $n \geq 0$

$$L(x, n) = \{g \in T_x \ : \ n - \Delta < \nu(g) \leq n\},$$

and

$$u(x, n) = \sum_{g \in L(x,n)} e^{\psi(g)}.$$

Lemma 6.3. *For every $k > 0$ there exist positive constants c_1, c_2 such that*

$$c_1 u(x, n) \leq u(x, n + k) \leq c_2 u(x, n)$$

for all $x \in X$ and $n > 0$.

Proof. For every element $g = g_t \cdots g_1 \in L(x, n + k)$, where $g_i \in S$, there exists an index $j < t$ such that $h = g_j \cdots g_1 \in L(x, n)$. Choose one such h for every $g \in L(x, n + k)$ and denote it $\varphi_1(g)$. There is a uniform upper bound (depending on k but not on x or n) on the distance from g to $\varphi_1(g)$ in the Cayley graph, hence

there is a uniform upper bound q on $|\varphi_1^{-1}(h)|$, and a uniform upper bound c_0 on $|\psi(g) - \psi(\varphi_1(g))|$. It follows that

$$u(x, n+k) = \sum_{g\in L(x,n+k)} e^{\psi(g)} \le e^{c_0} \sum_{g\in L(x,n+k)} e^{\psi(\varphi_1(g))} \le$$

$$qe^{c_0} \sum_{h\in L(x,n)} e^{\psi(h)} = qe^{c_0} u(x,n).$$

On the other hand for every element $h = g_l \cdots g_1 \in L(x, n)$ there exists $\varphi_2(h) \in L(x, n+k)$ of the form $g_r \cdots g_{l+1} g_l \cdots g_1$ where $r \ge l$. Again, there is a uniform upper bound on the distance between h and $\varphi_2(h)$. Hence, there are uniform upper bounds on $|\psi(g) - \psi(\varphi_2(h))|$ and on $|\varphi_2^{-1}(h)|$, which implies an inequality of the form $u(x, n) \le c_1^{-1} u(x, n+k)$. □

Lemma 6.4. *There exists a constant $c \ge 1$ such that for any two points $x, y \in X$ we have*

$$c^{-1} u(y, n) \le u(x, n) \le cu(y, n)$$

for all $n \ge 0$.

Proof. The sets of values of the transformations R_g^h in general do not belong to $G|_X$, since the ranges of the elements $F \in \mathcal{S}$ do not belong to X. However, the ranges of $F \in \mathcal{S}$ belong to a compact set X' containing X, and since X is a topological transversal, there exists a compact set $Q \subset \mathfrak{G}$ such that for every $f \in R_g^h$ there exists $q_f \in Q$ such that $q_f f \in \mathfrak{G}|_X$. Choose such q_f for every $f \in T_g$, and define

$$\widetilde{R}_g^h(f) = q_f f. \tag{8}$$

Note that both transformations R_g^h and $\widetilde{R}_g^h$ have the same continuous extension onto $\mathcal{T}_g$.

Lemma 6.5. *There is a constant $N > 0$ such that for every pair $x, y \in X$ there exists $g \in T_x$ such that R_y^g is defined, $\nu(g) < N$, and $\widetilde{R}_y^g(T_y) \subset T_x$.*

Proof. Let δ_0 be as in Lemma 3.1 Consider a finite covering $\{W_i\}_{i\in I}$ of the set X by open subsets of diameter less than δ_0.

Let a point $\zeta_x \in \mathcal{T}_x$ and a number r_0 satisfy the conditions of Proposition 6.2. Then the r_0-neighborhood of $\zeta_x = \cdots h_2 h_1$ in $\overline{\mathfrak{G}_x^X}$ is contained in $\overline{T_x}$. By minimality of $\mathfrak{G}$, for every $i \in I$ the set of elements $g \in \mathfrak{G}_x^X$ such that $\mathsf{t}(g) \in W_i$ is an Ξ-net (with respect to the usual combinatorial metric on the Cayley graph) in $\mathfrak{G}(x, S)$ for some fixed $\Xi > 0$ (not depending on x and i).

If $q \in \mathfrak{G}|_X$ is an element of length at most Ξ such that $\mathsf{o}(q) = \mathsf{t}(h_l \cdots h_1)$, and D is an upper bound on the values of $|\nu|$ on elements of S, then the value of ν on a geodesic path connecting $h_l \cdots h_1$ with $qh_l \cdots h_1$ in the Cayley graph $\mathfrak{G}(x, S)$ is bounded below by $\nu(h_l \cdots h_1) - (D+\eta)\Xi > l\eta - (D+\eta)\Xi$. It follows that there exists l_0 not depending on x such that for every $l \ge l_0$ the (combinatorial) Ξ-neighborhood of $h_l \cdots h_1$ in the Cayley graph $\mathfrak{G}(x, S)$ is contained in the (hyperbolic) $r_0/2$-neighborhood of ζ_x.

If l is big enough, g belongs to the combinatorial Ξ-neighborhood of $h_l \cdots h_1$, and T_y^g is defined, then by (6), the hyperbolic diameter of the set of values of $\widetilde{T}_y^g$ is less than $r_0/2$.

It follows that there exists a constant N such that for every $i \in I$ there exists $g_i \in T_x$ such that $\mathsf{t}(g) \in W_i$, $\nu(g) < N$, the set of values of $\widetilde{T}_y^g$ for every $y \in W_i$ is contained in the r_0-neighborhood of ζ_x, hence is contained in T_x. □

The set of elements g satisfying the conditions of Lemma 6.5 is contained in a compact set of the form $\bigcup_{k=0}^n S^k$ for some n not depending on x and y. It follows, by dualizability of ν and ψ, that the differences $|\nu(\widetilde{R}_y^g(h))-\nu(h)|$ and $|\psi(\widetilde{R}_y^g(h))-\psi(h)|$ are uniformly bounded.

The map R_y^g is injective by Lemma 3.1. Consequently, the cardinalities of the sets $(\widetilde{R}_y^g)^{-1}(h)$ are uniformly bounded.

Consequently, using Lemma 6.3 we have an estimate of the form $u(y,n) \le c \cdot u(x,n)$. □

Proposition 6.6. *There exists a constant c_2 such that for every $x \in X_0$ and any positive numbers n_1, n_2 we have*

$$c_2^{-1}u(x,n_1)u(x,n_2) \le u(x,n_1+n_2) \le c_2 u(x,n_1)v(x,n_2).$$

Proof. There exists a constant q_1 such that every element $g \in L(x, n_1+n_2)$ can be decomposed into a product $g = g_1g_2$ such that $\nu(g_1)$ and $\nu(g_2)$ belong to the intervals $[n_1-\Delta, n_1+\Delta]$ and $[n_2-\Delta, n_2+\Delta]$ respectively, at least in one and at most in q_1 ways.

It follows (using Lemmas 6.4 and 6.3) that $u(x,n_1+n_2) \le k_1 \cdot u(x,n_1)u(x,n_2)$ for some constant k_1.

On the other hand, for any pair $g_1 \in L(x,n_1)$ and $g_2 \in L(\mathsf{t}(g_1), n_2)$ we have $n_1+n_2-2\Delta-\eta < \nu(g_2g_1) < n_1+n_2+\eta$. There exists a constant $q_2 > 1$ such that there exist at most q_2 pairs $h_1 \in L(x,n_1)$ and $h_2 \in L(\mathsf{t}(h_1), n_2)$ such that $h_2h_1 = g_2g_1$. Hence (again using Lemmas 6.4 and 6.3) we have $u(x,n_1+n_2) \ge c_2^{-1}u(x,n_1)u(x,n_2)$ for some $c_2 > 1$. □

The following lemma is Exercise 99 in [23] (next after a more famous problem on sub-additive sequences).

Lemma 6.7. *Let a_n, $n \ge 1$, be a sequence of real numbers such that $a_{n_1}+a_{n_2}-1 \le a_{n_1+n_2} \le a_{n_1}+a_{n_2}+1$ for all n_1 and n_2. Then the limit $\rho = \lim_{n\to\infty} a_n/n$ exists and $n\rho - 1 \le a_n \le n\rho+1$ for all n.*

Let now c_2 be as in Proposition 6.6. Define the sequence

$$\alpha_n = \ln(u(x,n))/\ln c_2.$$

Then Proposition 6.6 implies that for any n_1, n_2, we have

$$\alpha_{n_1}+\alpha_{n_2}-1 \le a_{n_1+n_2} \le \alpha_{n_1}+\alpha_{n_2}+1,$$

and by Lemma 6.7 it follows that the limit $\lim_{n\to\infty} \alpha_n/n = \rho$ exists and $n\rho - 1 \le \alpha_n \le n\rho + 1$ for all n.

Consequently there exist constants β and $k > 1$ such that

$$k^{-1}e^{n\beta} \le u(x,n) \le ke^{n\beta} \tag{9}$$

for all $n > 0$ and $x \in X$.

Let now C be any compact neighborhood in $\overline{\mathfrak{G}_x^X}$ of a point of $\partial\mathfrak{G}_x$. Then by compactness, C can be covered by a finite number of sets of the form $\overline{T_g}$, which gives us an upper bound of the form $k_2e^{n\beta}$ for $\sum_{g\in C\cap\mathfrak{G}_x, n-\Delta\le\nu(g)\le n} e^{\psi(g)}$. On the

other hand, since the collection $\overline{T_g}$ for $g \in \mathfrak{G}_x$ is a basis of neighborhoods of points of $\partial\mathfrak{G}_x$, there exists a subset of C of the form $\overline{T_g}$, which gives us a lower bound finishing the proof of the theorem. □

The following proposition is a direct corollary of Theorem 6.1.

Proposition 6.8. *Let ν and ψ be a Busemann and a dualizable quasi-cocycles on $\mathfrak{G}$, respectively. Let f be a continuous function of compact support on $\overline{\mathfrak{G}_x^X}$ not identically equal to zero on $\partial\mathfrak{G}_x$. Consider the series*

$$\mathcal{P}_{f,\nu,\psi}(s) = \sum_{g\in\mathfrak{G}_x^X} f(g)e^{-s\nu(g)+\psi(g)}.$$

If β is pressure of ψ relative to ν, then the series $\mathcal{P}_{f,\nu,\psi}(s)$ diverges for $s \le \beta$ and converges for $s > \beta$.

6.2. **Entropy of hyperbolic groupoids and Smale quasi-flows.**

Definition 6.3. Pressure of the zero cocycle $\psi(g) = 0$ relative to the Busemann cocycle ν is called the *entropy* of the graded groupoid $(\mathfrak{G}, \nu)$ and is denoted $h(\mathfrak{G}, \nu)$, or just $h(\nu)$.

Proposition 6.9. *Entropy of a hyperbolic groupoid is positive and*

$$h(\nu) = \lim_{n\to\infty} \frac{\ln |\{g \in T_x \ : \ \nu(g) \le n\}|}{n}.$$

for every $x \in X$.

Proof. It is enough to prove that entropy is positive, i.e., that sequence $u(x, n)$ from the proof of Theorem 6.1 is unbounded.

Suppose, by contradiction that $u(x, n) < m$ for every n. Since every path $g_1, g_2g_1, g_3g_2g_1, \ldots$ connecting x to a point $\xi \in \mathcal{T}_x$ (where $g_i \in S$) intersects each of the sets $L(x, n)$, and any two such paths which have infinite intersection converge to the same point of $\mathcal{T}_x$, we get that $|\mathcal{T}_x| < m$, in particular, that $\mathfrak{G}^{(0)}$ has isolated points. Then we can find a singleton that is a $\mathfrak{G}$-transversal (by minimality of $\mathfrak{G}$). But groupoid of germs of a pseudogroup acting on a single point can not satisfy the conditions of Definition 2.4. □

Let (S, X) be a generating pair of $\mathfrak{G}$ satisfying Definition 2.4. For a finite subset $N \subset X$ denote by $v(N, n)$ cardinality of the set of elements $g \in \mathfrak{G}$ such that g is a product of elements of S, $\nu(g) \le n$, and $\mathsf{o}(g) \in N$.

Lemma 6.10. *Supremum of the number*

$$\beta_N = \lim_{n\to\infty} \frac{\ln v(N, n)}{n}$$

over finite subsets $N \subset X$ is attained and is equal to the entropy $h(\nu)$.

Proof. Since S can be embedded into a generating set satisfying Proposition 2.2, β_N is not greater than the entropy. It is also obvious that β_N does not decrease when we increase the set N. Consequently, it is enough to show that there exists N such that β_N is equal to the entropy.

Let $\xi \in \partial\mathfrak{G}_x$, and let C be a compact neighborhood of ξ in $\overline{\mathfrak{G}_x^X}$. Then there exists a finite set $A \subset \mathfrak{G}_x^X$ such that C is included in the set of products $\ldots g_2g_1g$ for $g_i \in S$ and $g \in A$. It follows then that β_N for $N = \mathsf{t}(A)$ is not less than the entropy. □

Theorem 6.11. *Entropies of* $(\mathfrak{G}, \nu)$ *and* $(\mathfrak{G}, \nu)^\top$ *are equal.*

Proof. Let (S, X) be a generating pair of $\partial\mathfrak{G} \rtimes \mathfrak{G}$ and let $\mathcal{S}$ be a covering of S by rectangles satisfying the conditions of [21, Definition 6.1].

Choosing the elements of the covering $\mathcal{S}$ small enough, we may assume that there exists $c > 0$ such that for every non-empty product $U_1 \cdots U_n$ of elements of $\mathcal{S}$ the values of

$$\widetilde{\nu}(g), \widetilde{\nu}(h), \nu(\mathsf{P}_+(g)), \nu^\top(\mathsf{P}_-(g))$$

differ from each other not more than by c for all $g, h \in U_1 \cdots U_n$ (see Corollary 5.5).

Let ϵ be a Lebesgue number of the covering $\mathcal{S}$, and let $\mathcal{R}$ be a covering of X by a finite number of open rectangles of diameter less than ϵ.

Since the Smale quasi-flow $\partial\mathfrak{G} \rtimes \mathfrak{G}$ is locally diagonal (see Proposition 4.8), we may assume that for any two rectangles $R_1, R_2 \in \mathcal{R}$ and elements $g_1, g_2 \in \mathfrak{H}$ such that $\mathsf{o}(g_i) \in R_1$ and $\mathsf{t}(g_i) \in R_2$ equalities $\mathsf{P}_+(g_1) = \mathsf{P}_+(g_2)$ or $\mathsf{P}_-(g_1) = \mathsf{P}_-(g_2)$ imply $g_1 = g_2$. Consider localization $\mathfrak{H}$ of $\partial\mathfrak{G} \rtimes \mathfrak{G}$ onto $\mathcal{R}$.

Let $N_+ \subset \mathsf{P}_+(\mathfrak{H}^{(0)})$ and $N_- \subset \mathsf{P}_-(\mathfrak{H}^{(0)})$ be finite subsets such that β_{N_+} and β_{N_-} are equal to the entropies of $\mathsf{P}_+(\mathfrak{H})$ and $\mathsf{P}_-(\mathfrak{H})$. We may assume that N_+ and N_- have non-empty intersections with $\mathsf{P}_+(R)$ and $\mathsf{P}_-(R)$, respectively, for every $R \in \mathcal{R}$.

Let $V(N_i, n)$, for $i \in \{+, -\}$, be the set of elements $h_i \in \mathsf{P}_i(\mathfrak{H})$ such that h_i is a product of elements of $\mathsf{P}_i(S)$, $\nu(h_i) \le n$, and $\mathsf{o}(h_i) \in N_i$. Then $\beta_{N_i} = \lim_{n\to\infty} \ln |V(N_i, n)| / \ln n$. We will denote $V(N_i) = \bigcup_{n\ge 1} V(N_i, n)$.

We will say that $g_+ \in V(N_+)$ and $g_- \in V(N_-)$ are *related* if there exists $h \in \mathfrak{H}$ such that h is a product of elements of S (more precisely of their copies in the localization), $\mathsf{P}_+(h) = g_+$, and $\mathsf{P}_-(h) = g_-$.

Suppose that $g_+ \in V(N_+, n)$ is equal to a product $\mathsf{P}_+(s_1) \cdots \mathsf{P}_+(s_k)$ of elements of $\mathsf{P}_+(S)$. Let $R_1, R_2 \in \mathcal{R}$ be such that $\mathsf{o}(s_k) \in R_1$ and $\mathsf{t}(s_1) \in R_2$. Let $U_i \in \mathcal{S}$ be such that s_i is ϵ-contained in U_i. Then $\mathsf{P}_+(\mathsf{o}(U_1 \cdots U_n) \cap R_1) = \mathsf{P}_+(R_1)$, and $\mathsf{P}_-(\mathsf{t}(U_1 \cdots U_n) \cap R_2) = \mathsf{P}_-(R_2)$. For any $x_- \in N_-$ such that $x_- \in \mathsf{P}_-(R_2)$ we can find germs $r_1, \ldots, r_k$ of $U_1, \cdots, U_k$ such that $\mathsf{P}_+(s_1) = \mathsf{P}_+(r_1)$, the product $r_1 \cdots r_k$ is defined, and $\mathsf{P}_-(\mathsf{t}(r_1)) = x_2$. Consequently, every element $g_+ \in V(N_1, n)$ is related to an element $g_- \in V(N_-, n + c)$. By the same argument, every element $g_- \in V(N_-, n)$ is related to an element $g_+ \in V(N_1, n + c)$.

Note that by local diagonality, an element of $V(N_+)$ can not be related to more than $|N_-|$ elements of $V(N_-)$, and similarly, an element of $V(N_-)$ can not be related to more than $|N_+|$ elements of $V(N_+)$.

Consequently, $|V(N_1, n)| \le |N_1| \cdot |V(N_2, n+2c)|$, and $|V(N_2, n)| \le |N_2| \cdot |V(N_1, n+2c)|$, which implies that $\beta_{N_1} = \beta_{N_2}$. $\square$

One can prove in a similar way that pressure of a cocycle ψ relative to ν is equal to pressure of $\psi^\top$ relative to $\nu^\top$.

7. Quasi-conformal measures

7.1. **Definition and basic properties.** Let $\widetilde{\mathfrak{G}}$ be a pseudogroup acting on a space $\mathfrak{G}^{(0)}$. A Radon measure μ on $\mathfrak{G}^{(0)}$ is *quasi-invariant* if for every $F \in \widetilde{\mathfrak{G}}$ and every $A \subset \mathsf{o}(F)$ such that $\mu(A) = 0$ we have $\mu(F(A)) = 0$.

If μ is quasi-invariant with respect to $\widetilde{\mathfrak{G}}$, then for every $F \in \widetilde{\mathfrak{G}}$ we have the corresponding Radon-Nicodim derivative

$$\rho_\mu(F, x) = \frac{dF^*\mu}{d\mu}(x),$$

where $x \in \mathsf{o}(F)$ and $F^*\mu$ is the pull-back of μ by F. Note that $\rho_\mu(F, x)$ depends only on the germ $(F, x) \in \mathfrak{G}$.

Integrating the counting measure on $\mathfrak{G}_x$ by μ we get a measure μ_{o} on $\mathfrak{G}$ given by the formula

$$\int f(g)\, d\mu_{\mathsf{o}}(g) = \int \sum_{g \in \mathfrak{G}_x} f(g)\, d\mu(\mathsf{o}(g)),$$

where $f : \mathfrak{G} \longrightarrow \mathbb{R}$ is a compactly supported continuous function. Similarly, we have a measure μ_{t} on $\mathfrak{G}$ given by

$$\int f(g)\, d\mu_{\mathsf{t}}(g) = \int \sum_{g \in \mathfrak{G}^x} f(g)\, d\mu(\mathsf{t}(g)).$$

Quasi-invariance of μ is equivalent to absolute continuity of μ_{o} and μ_{t} with respect to each other. In particular, if μ is quasi-invariant, then we have a well defined notion of null sets in $\mathfrak{G}$ with respect to μ (i.e., with respect to μ_{o} or μ_{t}). The Radon-Nicodim derivative $\rho_\mu(g)$ is equal to the Radon-Nicodim derivative $\frac{d\mu_{\mathsf{t}}}{d\mu_{\mathsf{o}}}(g)$.

It is easy to see that the map $\rho_\mu : \mathfrak{G} \longrightarrow \mathbb{R}_{>0}$ satisfies the multiplicative cocycle condition

$$\rho_\mu(g_1 g_2) = \rho_\mu(g_1)\rho_\mu(g_2)$$

for μ-almost all composable pairs.

Definition 7.1. Let $(\mathfrak{G}, \nu)$ be a graded hyperbolic groupoid. A Radon measure measure μ on $\mathfrak{G}^{(0)}$ is said to be *($\mathfrak{G}$)-quasi-conformal* if it is $\mathfrak{G}$-quasi-invariant, and there exists $\beta > 0$ such that

$$\rho_\mu(g) \asymp e^{-\beta \cdot \nu(g)}$$

for all $g \in \mathfrak{G}$. The number β is called the *exponent* of the quasi-conformal measure.

Note that quasi-conformality of the measure does not depend on the choice of the quasi-cocycle (i.e., if a measure is quasi-conformal with respect to one quasi-cocycle, then it is quasi-conformal with respect to any strongly equivalent quasi-cocycle).

Proposition 7.1. *Let $(\mathfrak{G}, \nu)$ be a minimal graded hyperbolic groupoid, where $\nu : \mathfrak{G} \longrightarrow \mathbb{R}$ is everywhere defined. Let $X_0 \subset \mathfrak{G}^{(0)}$ be an open subset, and suppose that there exists a $\mathfrak{G}|_{X_0}$-quasi-conformal measure μ_0 on $\mathfrak{G}|_{X_0}$. Then there exists a $\mathfrak{G}$-quasi-conformal measure μ on $\mathfrak{G}^{(0)}$ of the same exponent as μ_0.*

Proof. Since every open subset of $\mathfrak{G}^{(0)}$ is a $\mathfrak{G}$-transversal, for every $x \in \mathfrak{G}^{(0)}$ there exists $g \in \mathfrak{G}$ such that $\mathsf{t}(g) = x$ and $\mathsf{o}(g) \in X_0$. Hence, there exists a set $\mathcal{U} \subset \widetilde{\mathfrak{G}}$ such that $\{\mathsf{t}(U) \,:\, U \in \mathcal{U}\}$ is a covering of $\mathfrak{G}^{(0)}$, and $\mathsf{o}(U) \subset X_0$ for all $U \in \mathcal{U}$.

Let $\{\phi_U\}_{U \in \mathcal{U}}$ be a partition of unity, where $\varphi_U : \mathfrak{G}^{(0)} \longrightarrow [0, 1]$ is a continuous non-negative (possibly zero) function with compact support contained in $\mathsf{t}(U)$.

Define then a measure μ on $\mathfrak{G}^{(0)}$ by the formula

$$\int f(x)\, d\mu(x) = \sum_{U \in \mathcal{U}} \int f(U(y))\varphi_U(U(y)) e^{-\beta\nu(U,y)}\, d\mu_0(y),$$

where β is the exponent of μ_0, and $f : \mathfrak{G}^{(0)} \longrightarrow \mathbb{R}$ is a continuous function of compact support.

Let $h \in \mathfrak{G}$ be such that $\mathsf{o}(h) \in X_0$, and let $x = \mathsf{t}(h)$. We have

$$\frac{dh^*\mu}{d\mu_0}(x) = \sum_{U\in\mathcal{U}} \varphi_U(x) e^{-\beta\nu(U,U^{-1}(x))} \frac{d(U^{-1}\circ h)^*(\mu_0)}{d\mu_0}(\mathsf{o}(h)) \asymp \sum_{U\in\mathcal{U}} \varphi_U(x) e^{-\beta\nu(U,U^{-1}(x))} e^{-\beta\nu(U^{-1}h)} \asymp \sum_{U\in\mathcal{U}} \varphi_U(x) e^{-\beta\nu(h)} = e^{-\beta\nu(h^{-1})}.$$

It follows that for every $g \in \mathfrak{G}$ we have $\frac{dg^*(\mu)}{d\mu} \asymp e^{-\beta\nu(g)}$, i.e., that μ is quasi-conformal. □

Corollary 7.2. *Let $(\mathfrak{G}, \nu)$ be a minimal graded hyperbolic groupoid. If there exists a quasi-conformal measure on a graded groupoid equivalent to $(\mathfrak{G}, \nu)$, then it exists on $(\mathfrak{G}, \nu)$.*

Proposition 7.3. *The exponent of a quasi-conformal measure is equal to the entropy of the groupoid $\mathfrak{G}$.*

Proof. We will prove this proposition for the dual groupoid $\mathfrak{G}^\top$. Let (S, X) be a generating pair of $\mathfrak{G}$ satisfying the conditions of Proposition 2.2. We will realize $\mathfrak{G}^\top$ as the groupoid $\mathfrak{d}\mathfrak{G}_x$ acting on the boundary $\partial\mathfrak{G}_x$ of a Cayley graph $\mathfrak{G}(x, S)$. Let μ be a $\mathfrak{G}^\top$-quasi-invariant measure on $\partial\mathfrak{G}_x$.

It follows from Lemma 6.5 that there exists a compact set Q such that for every $g, h \in \mathfrak{G}_x^X$ there exists $q \in Q$ such that $qh \in T_h$, R_g^{qh} is defined, and $\widetilde{R}_g^{qh}(T_g) \subset T_h$. Then $R_g^{qh}(\mathcal{T}_g) \subset \mathcal{T}_h$. The values of $\nu^\top$ on the germs of R_g^{qh} are equal (up to a uniformly bounded additive constant) to $\nu(h) - \nu(g)$ (see Proposition 3.2). It follows that there exists a constant $c > 1$ such that $\mu(\mathcal{T}_h) \geq \mu(R_g^{qh}(\mathcal{T}_g)) \geq c^{-1}e^{-\beta(\nu(h)-\nu(g))}\mu(\mathcal{T}_g)$, i.e., such that $e^{\beta\nu(h)}\mu(\mathcal{T}_h) \geq c^{-1}e^{\beta\nu(g)}\mu(\mathcal{T}_g)$. It follows that

$$\mu(\mathcal{T}_g) \asymp e^{-\beta\nu(g)} \tag{10}$$

for all g.

Let $u(x, n)$ be as in the proof of Theorem 6.1 (for $\psi = 0$). Then for every n we get a covering of $\mathcal{T}_x$ by at most $u(x, n)$ sets $\mathcal{T}_h$ such that $\mu(\mathcal{T}_h) \asymp e^{-\beta n}$, and there is a constant $k > 1$ such that we can find at least $k^{-1} \cdot u(x, n)$ disjoint subsets $\mathcal{T}_h$ such that $\mu(\mathcal{T}_h) \asymp e^{-\beta n}$. It follows that there exist a constant $c_0 > 1$ such that

$$c_0^{-1}u(x, n)e^{-\beta\nu(h)} \leq \mu(\mathcal{T}_x) \leq c_0 u(x, n)e^{-\beta\nu(h)}.$$

But $\mu(\mathcal{T}_x)$ is a constant, and $u(x, n) \asymp e^{\beta_0 n}$, where β_0 is entropy of $(\mathfrak{G}, \nu)$. Consequently, $\beta = \beta_0$. Theorem 6.11 now finishes the proof. □

Proposition 7.4. *Let μ be a quasi-conformal measure on an open transversal X_0 of a graded minimal hyperbolic groupoid $(\mathfrak{G}, \nu)$. Let $|\cdot|$ be a hyperbolic metric on X_0 of exponent α. Let $\beta = h(\nu)$. Let $X_1 \subset X_0$ be a compact topological transversal. Then for all $r > 0$ small enough and all $x \in X_1$ we have*

$$\mu(B(x, r)) \asymp r^{\beta/\alpha}.$$

Proof. It is enough to prove the proposition for any equivalent groupoid. Consequently, we can use duality, and prove the proposition for the groupoid $\mathfrak{G}^\top = \mathfrak{d}\mathfrak{G}_x$ instead of $\mathfrak{G}$.

Every ball $B(\xi, r) \subset \partial\mathfrak{G}_x$ is contained in the set of points $\zeta \in \partial\mathfrak{G}_x$ such that $\ell(x,y) > \ln r/\alpha - c$ for some constant $c > 0$. Recall that $\ell(x,y)$ is the minimal value of ν along a geodesic path in a Cayley graph of $\mathfrak{G}$ connecting ξ to ζ. It follows that the ball $B(\xi, r)$ can be covered by a bounded number (not depending on ξ and r) of sets of the form $\mathcal{T}_h$, such that $\nu(h) \doteq -\ln r/\alpha$.

On the other hand, moving h along the geodesic converging to ξ we can find a set $\mathcal{T}_h \subset B(\xi, r)$ such that $\nu(h) \doteq -\ln r/\alpha$.

By (10), $\mu(\mathcal{T}_h) \asymp e^{-\beta\nu(h)}$. We get then the necessary estimates from both sides to show that $\mu(B(\xi, r)) \asymp e^{\beta \ln r/\alpha} = r^{\beta/\alpha}$. $\square$

Corollary 7.5. *Every quasi-conformal measure on $\mathfrak{G}^{(0)}$ is equivalent to the Hausdorff measure of the hyperbolic metric of dimension β/α, where $\beta = h(\nu)$, and α is exponent of the hyperbolic metric. In particular, any two quasi-conformal measures are equivalent.*

7.2. **Existence of quasi-conformal measures.** We will apply here the standard construction of a Patterson-Sullivan measure on boundaries of hyperbolic graphs (see, for example [4]) to show existence of quasi-conformal measures.

Let $(\mathfrak{G}, \nu)$ be a graded hyperbolic groupoid, and let $\mathfrak{G}(x, S)$ be its Cayley graph. Denote by $\beta = h(\nu)$.

Lemma 7.6. *Let $f : \overline{\mathfrak{G}_x^X} \longrightarrow \mathbb{C}$ be a continuous function of compact support. Conciser the series*

$$\mathcal{P}(s) = \sum_{g \in \mathfrak{G}_x^X} f(g) e^{-s\nu(g)}.$$

There exists a constant $c > 0$ such that $|\mathcal{P}(s)| \le c(1 - e^{\beta - s})^{-1}$ for all $s > \beta$ that are sufficiently close to β.

Recall, that by Proposition 6.8, the series $\mathcal{P}(s)$ converges for all $s > \beta$.

Proof. Let C be the support of f. By Theorem 6.1, there exist positive integers Δ, k, and n_0 such that

$$|\{g \in C \cap \mathfrak{G}_x \ : \ n - \Delta \le \nu(g) \le n\}| \le k e^{\beta n}$$

for all $n \ge n_0$. Denote $L(n) = \{g \in C \cap \mathfrak{G}_x \ : \ n - \Delta \le \nu(g) \le n\}$. Let $c_1 = \sup |f|$. We have

$$\begin{aligned}
|\mathcal{P}(s)| \le \sum_{g \in C \cap \mathfrak{G}_x} c e^{-s\nu(g)} \le \\
c_1 \sum_{n < n_0} |L(n)| e^{-s(n-\Delta)} + c_1 \sum_{n \ge n_0} |L(n)| e^{-s(n-\Delta)} \le \\
c_1 \sum_{n < n_0} |L(n)| e^{-s(n-\Delta)} + c_1 k e^{s\Delta} \sum_{n \ge 0} e^{(\beta - s)n} = \\
c_1 \sum_{n < n_0} |L(n)| e^{-s(n-\Delta)} + c_1 k e^{s\Delta} (1 - e^{\beta - s})^{-1}.
\end{aligned}$$

There exists only a finite number of elements of C such that $\nu(g) < n_0$, consequently the first summand is continuous for all s, and hence its product with $(1 - e^{\beta - s})$ goes to zero as $s \to \beta$. $\square$

Let measure μ_s on $\overline{\mathfrak{G}_x^X}$ for $s > \beta$ be given by

$$\int f\, d\mu_s = (1 - e^{\beta - s}) \sum f(g) e^{-s\nu(g)}. \tag{11}$$

for all continuous functions f on $\overline{\mathfrak{G}_x^X}$ of compact support.

Proposition 7.7. *There exists a sequence $s_k \to \beta+$ such that μ_{s_k} is weakly converging to a measure μ. The limit measure μ is supported on $\partial\mathfrak{G}_x$ and is quasi-conformal with respect to $(\mathfrak{G}^\top, \nu^\top)$.*

Proof. By Uniform Boundedness Principle, the set $\{\mu_s \;:\; s > \beta\}$ is bounded in the space dual to the space of continuous compactly supported functions on $\overline{\mathfrak{G}_x^X}$. Hence, by Banach-Alaoglu Theorem, there exists a sequence s_k such that $s_k \to \beta+$ and μ_{s_k} is weakly converging to a measure on $\overline{\mathfrak{G}_x^X}$.

Since for every $g \in \mathfrak{G}_x^X$ (i.e., for any point $g \in \overline{\mathfrak{G}_x^X} \setminus \partial\mathfrak{G}_x$) we have $(1 - e^{\beta - s})e^{-s\nu(g)} \to 0$ as $s \to \beta$, the support of μ is contained in $\partial\mathfrak{G}_x$. Suppose that $f : \overline{\mathfrak{G}_x^X} \longrightarrow \mathbb{R}$ is a continuous non-negative compactly supported function, and let $f(\xi) > 0$ for a point $\xi \in \partial\mathfrak{G}_x$. Then for any positive number p less than $f(\xi)$ there exists a compact neighborhood C_0 of ξ in $\overline{\mathfrak{G}_x^X}$ such that $f(x) > p$ for all $x \in C_0$. Let Δ be as in Theorem 6.1. Denote by $L(n)$ the set of elements $g \in C_0$ such that $n - \Delta \le \nu(g) \le n$. Then the size of the set $L(n)$ is bounded below by $k_0 e^{\beta n}$ for some $k_0 > 0$ and for all $n \ge n_1$ for some k_1. Every point of C_0 belongs to at most $\Delta + 1$ sets $L(n)$. Consequently,

$$\int f\, \mu_s = (1 - e^{\beta - s}) \sum f(g) e^{-s\nu(g)} \ge (1 - e^{\beta - s}) \sum_{g \in C_0} p e^{-s\nu(g)} \ge$$
$$(1 - e^{\beta - s}) k_0 p (1 + \Delta)^{-1} \sum_{n \ge n_1} e^{n\beta} e^{-sn} =$$
$$(1 - e^{\beta - s}) k_0 p \Delta^{-1} e^{(\beta - s)n_1} (1 - e^{(\beta - s)})^{-1} = k_0 p \Delta^{-1} e^{(\beta - s)n_1},$$

hence $\int f\, d\mu > 0$, and support of μ coincides with $\partial\mathfrak{G}_x$.

Let $\widetilde{R}_h^g$ be partial transformations of $\overline{\mathfrak{G}_x^X}$ defined by (8). The values of $\nu^\top$ on germs of $\widetilde{R}_h^g$ on $\partial\mathfrak{G}_x$ differ from $\nu(g) - \nu(h)$ by a uniformly bounded constant not depending on g or h see (6).

Taking δ_0 in the definition of maps R_h^g sufficiently small, and using Corollary 5.5, we get an estimate $\nu(\widetilde{R}_h^g(x)) \doteq \nu(g) - \nu(h)$ for all $x \in T_h$, whence

$$e^{-s\nu(\widetilde{R}_h^g(x))} \asymp e^{-s\nu(x)} e^{-s(\nu(g) - \nu(h))} \asymp e^{-s\nu(x)} e^{-s\nu^\top(\gamma)}$$

for every $x \in T_h$ and for any germ γ of R_h^g on $\mathcal{T}_h$. It follows that for every subset $A \subset T_h$ we have

$$\mu_s(\widetilde{R}_h^g(A)) \asymp e^{-s\nu^\top(\gamma)} \mu_s(A),$$

where γ is any germ of R_h^g on $\mathcal{T}_h$. Consequently, the measure μ is $(\mathfrak{G}^\top, \nu^\top)$-quasi-conformal. $\square$

8. Continuous cocycles

8.1. **General definitions.**

Definition 8.1. Let $\mathfrak{G}$ be a topological groupoid. A map $\nu : \mathfrak{G} \longrightarrow \mathbb{R}$ is a *continuous cocycle* if it is continuous and $\nu(gh) = \nu(g) + \nu(h)$ for all $(g, h) \in \mathfrak{G}^{(2)}$.

An *orbispace* is an equivalence class of a proper groupoid of germs. A groupoid $\mathfrak{G}$ is said to be *proper* if the map $\mathsf{o}\times\mathsf{t}:\mathfrak{G}\longrightarrow\mathfrak{G}^{(0)}\times\mathfrak{G}^{(0)}$ is proper, i.e., if for this map preimages of compact sets are compact.

Note that if a groupoid of germs $\mathfrak{G}$ is proper and principal (i.e., if all isotropy groups $\mathfrak{G}_x^x$ are trivial), then it is equivalent to the trivial groupoid (a groupoid without non-unit elements) on the space of $\mathfrak{G}$-orbits.

A *flow* (resp. a $\mathbb{Z}$*-action*) on an orbispace is an equivalence class (in the sense of [20]) of a proper groupoid of germs $\mathfrak{G}$ together with an action F_t of $\mathbb{R}$ (resp. of $\mathbb{Z}$) on $\mathfrak{G}^{(0)}$ such that for every $g\in\mathfrak{G}$ and every $t\in\mathbb{R}$ (resp. $\in\mathbb{Z}$) there exists a unique element $g_t\in\mathfrak{G}$ such that the germs $F_t\circ g$ and $g_t\circ F_t$ are equal. The corresponding topological flow is given by the action of $\mathbb{R}$ (resp. of $\mathbb{Z}$) on the space of orbits of $\mathfrak{G}$.

Proposition 8.1. *Every Smale quasi-flow $(\mathfrak{H},\nu)$ with a continuous cocycle ν is equivalent to a flow on an orbispace.*

Proof. Consider the space $\mathfrak{H}^{(0)}\times\mathbb{R}$ and the action of $\mathfrak{H}$ on it defined by the rule:

$$(g,t)\cdot h=(gh,t+\nu(h)).$$

The obvious action of $\mathbb{R}$ on $\mathfrak{H}^{(0)}\times\mathbb{R}$ commutes with the defined action of $\mathfrak{H}$. We get then commuting actions on $\mathfrak{H}^{(0}\times\mathbb{R}$ of $\mathfrak{H}$ and of the groupoid $\widehat{\mathfrak{H}}$ generated by the action of $\mathfrak{H}$ and $\mathbb{R}$. It follows from [21, Theorem 6.5] and condition (5) of [21, Definition 6.1] that these actions are proper. They are also obviously free. It follows that they define equivalence between $\mathfrak{H}$ and $\widehat{\mathfrak{H}}$ in the sense of [20].

It also follows from properness of the action of $\mathfrak{H}$ on $\mathfrak{H}^{(0)}\times\mathbb{R}$ that the groupoid of this action together with the natural action of $\mathbb{R}$ define an orbispace flow. □

Definition 8.2. Two continuous cocycles ν_1 and ν_2 on $\mathfrak{G}$ are *co-homologous* if there exists a continuous function $\phi:\mathfrak{G}^{(0)}\longrightarrow\mathbb{R}$ such that

$$\nu_1(g)-\nu_2(g)=\phi(\mathsf{t}(g))-\phi(\mathsf{o}(g))$$

for all $g\in\mathfrak{G}$.

Let $\nu:\mathfrak{G}\longrightarrow\mathbb{R}$ be a continuous cocycle, let $f:Y\longrightarrow\mathfrak{G}^{(0)}$ be an étale map, and let $\mathfrak{G}|_f$ be the corresponding localization. Then *lift* ν_f of ν to $\mathfrak{G}|_f$ is given by $\nu_f(x,g,y)=\nu(g)$, where (x,g,y) is a lift of g. It is easy to see that ν_f is a continuous and well defined cocycle.

Recall that two groupoids are equivalent if and only if they have isomorphic localization (see a remark just after Definition 2.1).

Definition 8.3. We say that two continuous cocycles $\nu_1:\mathfrak{G}_1\longrightarrow\mathbb{R}$ and $\nu_2:\mathfrak{G}_2\longrightarrow\mathbb{R}$ defined on equivalent groupoids are *continuously equivalent* if there exists a common localization of $\mathfrak{G}_1$ and $\mathfrak{G}_2$ such that the lifts of ν_1 and ν_2 to it are cohomologous.

It is not hard to see that two continuously equivalent graded groupoids define topologically conjugate flows. Choice of a particular graded groupoid in a continuous equivalence class correspond in some sense to the choice of a "generalized transversal" of the flow.

8.2. **Hölder continuous cocycles.**

Definition 8.4. Let $\mathfrak{G}$ be a groupoid of germs preserving Lipschitz class of a metric $|\cdot|$ on $\mathfrak{G}^{(0)}$. We say that a cocycle $\nu : \mathfrak{G} \longrightarrow \mathbb{R}$ is *Hölder continuous* if there exists $p > 0$ such that for every $g \in \mathfrak{G}$ there exists a neighborhood $U \in \widetilde{\mathfrak{G}}$ of g such that $\nu(U, x)$ is p-Hölder continuous as a function of x with respect to $|\cdot|$.

Proposition 8.2. *Let $\mathfrak{H}$ be a Smale quasi-flow and let $\psi : \mathfrak{H} \longrightarrow \mathbb{R}$ be a Hölder continuous cocycle. Then there exists an equivalent groupoid $\mathfrak{H}'$ and a Hölder continuous cocycle $\psi' : \mathfrak{H}' \longrightarrow \mathbb{R}$ continuously equivalent to ψ, and a unique, up to cohomology, Hölder continuous cocycle $\psi_+ : \mathsf{P}_+(\mathfrak{H}) \longrightarrow \mathbb{R}$ such that $\psi_+(\mathsf{P}_+(g)) = \psi'(g)$ for all $g \in \mathfrak{H}'$.*

Proof. Possibly passing to a localization, we assume that there exist sets $X, X_1, S, \mathcal{S}$ satisfying the conditions of Proposition 2.4. We will prove our proposition for restriction of the groupoid onto X.

Choose a point $x_B \in B^\circ$ for every rectangle $A \times B$. For a point $y \in A^\circ \times B^\circ$ choose a sequence $F_1, F_2, \ldots$ of elements of $\mathcal{S}^{-1}$ such that $\mathsf{P}_-(\mathsf{t}(F_n)) \subset \mathsf{P}_-(\mathsf{o}(F_{n+1}))$ and $y \in \mathsf{o}(F_n \cdots F_1)$ for all n. Let $y' = [y, x_B]$. Then $\{y, y'\} \subset \mathsf{o}(F_n \cdots F_1)$ for all n and $|F_n \cdots F_1(y) - F_n \cdots F_1(y')| \asymp \lambda^n |y - y'|$ for some $\lambda \in (0, 1)$. Define then

$$\phi(y) = \lim_{n\to\infty} \psi(F_n \cdots F_1, y) - \psi(F_n \cdots F_1, y') = \sum_{n=1}^{\infty} (\psi(F_n, F_{n-1} \cdots F_1(y)) - \psi(F_n, F_{n-1} \cdots F_1(y'))).$$

Note that $|\psi(F_n, F_{n-1} \cdots F_1(y)) - \psi(F_n, F_{n-1} \cdots F_1(y'))| \le c\lambda^{pn} |y - y'|^p$ for some $c, p > 0$, by the Hölder continuity of ψ. It follows that the limit exists.

Let $y_1, y_2 \in A^\circ \times B^\circ$ such that $\mathsf{P}_+(y_1) = \mathsf{P}_+(y_2)$. Then

$$\phi(y_1) - \phi(y_2) = \sum_{n=1}^{\infty} (\psi(F_n, F_{n-1} \cdots F_1(y_1)) - \psi(F_n, F_{n-1} \cdots F_1(y_2)));$$

and since $|\psi(F_n, F_{n-1} \cdots F_1(y_1)) - \psi(F_n, F_{n-1} \cdots F_1(y_2))| \le c\lambda^{pn} |y - y'|^p$, the function ϕ is p-Hölder continuous on each slice $\mathsf{P}_+^{-1}(x)$.

Let us show that ϕ does not depend on the choice of the sequence F_i. Let $F_i' \in \mathcal{S}$ be another sequence. Then there exist strictly increasing sequences n_i and m_i, a finite set $\mathcal{A}$ of relatively compact elements of $\widetilde{\mathfrak{H}}$, and a sequence $U_i \in \mathcal{A}$ such that

$$(U_i F_{n_i} \cdots F_1, y) = (F'_{m_i} \cdots F'_1, y), \quad (U_i F_{n_i} \cdots F_1, y') = (F'_{m_i} \cdots F'_1, y')$$

for all i.

Then

$$\psi(F'_{m_i} \cdots F'_1, y) - \psi(F'_{m_i} \cdots F'_1, y') = \psi(U_i F_{n_i} \cdots F_1, y) - \psi(U_i F_{n_i} \cdots F_1, y') =$$
$$\psi(U_i, F_{n_i} \cdots F_1(y)) - \psi(U_i, F_{n_i} \cdots F_1(y')) + \psi(F_{n_i} \cdots F_1, y) - \psi(F_{n_i} \cdots F_1, y')$$

Since $|F_{n_i} \cdots F_1(y) - F_{n_i} \cdots F_1(y')| \asymp \lambda^{n_i} |y - y'|$, the difference $\psi(U_i F_{n_i} \cdots F_1, y) - \psi(U_i F_{n_i} \cdots F_1, y')$ goes to zero by Hölder continuity of ψ. Consequently, the limit $\phi(y)$ defined in terms of F_i is the same as the one defined in terms of F_i'.

Lemma 8.3. *Define $\psi'(g) = \psi(g) - \phi(\mathsf{o}(g)) + \phi(\mathsf{t}(g))$. Then for any two $g_1, g_2 \in \mathfrak{H}$ such that $\mathsf{P}_+(g_1) = \mathsf{P}_+(g_2)$ we have $\psi'(g_1) = \psi'(g_2)$.*

Proof. Consider two sequences $F_1, F_2, \ldots$ and $G_1, G_2, \ldots$ of elements of $\mathcal{S}$ such that $\mathsf{P}_-(\mathsf{t}(F_n)) \subset \mathsf{P}_-(\mathsf{o}(F_{n+1}))$, $\mathsf{P}_-(\mathsf{t}(G_n)) \subset \mathsf{P}_-(\mathsf{o}(G_{n+1}))$; and $\mathsf{o}(g_1), \mathsf{o}(g_2) \in \mathsf{o}(F_1)$,

$\mathsf{t}(g_1), \mathsf{t}(g_2) \in \mathsf{o}(G_1)$. Let $\mathcal{A}$ be as above. Then there exist increasing sequences n_i and m_i, a sequence $U_i \in \mathcal{A}$ and $\epsilon > 0$ such that $G_{n_i} \cdots G_1 g_1 (F_{m_i} \cdots F_1)^{-1}$ is ϵ-contained in U_i. It follows that for all i big enough we have

$$F_{m_i} \cdots F_1(\mathsf{o}(g_2)) \in \mathsf{o}(U_i).$$

Then $\mathsf{P}_+(U_i, F_{m_i} \cdots F_1(\mathsf{o}(g_2))) = \mathsf{P}_+(G_{n_i} \cdots G_1 g_1 (F_{m_i} \cdots F_1)^{-1})$, as U_i is a rectangle. But $\mathsf{P}_+(G_{n_i} \cdots G_1 g_2 (F_{m_i} \cdots F_1)^{-1}) = \mathsf{P}_+(G_{n_i} \cdots G_1 g_2 (F_{m_i} \cdots F_1)^{-1})$. By local diagonality of $\mathfrak{H}$, if two elements of $\mathfrak{H}$ have the same source and equal projections, then they are equal (we assume that the rectangles $A^\circ \times B^\circ$ are small enough). Consequently, $G_{n_i} \cdots G_1 g_2 (F_{m_i} \cdots F_1)^{-1} \in U_i$. It follows then that $\psi(G_{n_i} \cdots G_1 g_1 (F_{m_i} \cdots F_1)^{-1}) - \psi(G_{n_i} \cdots G_1 g_2 (F_{m_i} \cdots F_1)^{-1}) \to 0$ as $i \to \infty$. We have

$$\begin{aligned}
&\psi'(g_1) - \psi'(g_2) = \\
&\quad \psi(g_1) - \psi(g_2) - \phi(\mathsf{o}(g_1)) + \phi(\mathsf{o}(g_2)) + \phi(\mathsf{t}(g_1)) - \phi(\mathsf{t}(g_2)) = \\
&\quad \psi(g_1) - \psi(g_2) + \lim_{i\to\infty} \psi(F_{m_i} \cdots F_1, \mathsf{o}(g_2)) - \psi(F_{m_i} \cdots F_1, \mathsf{o}(g_1)) + \\
&\quad \lim_{i\to\infty} \psi(G_{n_i} \cdots G_1, \mathsf{t}(g_1)) - \psi(G_{n_i} \cdots G_1, \mathsf{t}(g_2)) = \\
&\quad \lim_{i\to\infty} \psi(G_{n_i} \cdots G_1 g_1 (F_{m_i} \cdots F_1)^{-1}) - \psi(G_{n_i} \cdots G_1 g_2 (F_{m_i} \cdots F_1)^{-1}) = 0.
\end{aligned}$$

$\square$

Let us show that ϕ is Hölder continuous, which will imply Hölder continuity of ψ'. Let ϵ be a Lebesgue number of the covering $\mathcal{S}$ of S. Suppose that F is Λ-Lipschitz for every $F \in \mathcal{S}$. If $|y_1 - y_2| < \epsilon\Lambda^{-k}$, then there exists a sequence $F_1, F_2, \ldots, F_k \in \mathcal{S}^{-1}$ such that $\mathsf{P}_-(\mathsf{t}(F_n)) \subset \mathsf{P}_-(\mathsf{o}(F_{n+1}))$ for all $1 \le n \le k-1$, and $y_1, y_2 \in \mathsf{o}(F_k \cdots F_1)$.

Continuing the sequence $F_1, \ldots, F_k$ to two sequences F_i and F_i' for y_1 and y_2, respectively, and repeating the estimates in the proof of uniqueness of ϕ, we get an estimate $|\phi(y_1) - \phi(y_2)| < C\lambda^k$, which implies that ϕ is Hölder continuous.

It remains to show that ψ' can be projected onto the groupoid $\mathsf{P}_+(\mathfrak{H})$, i.e., that there exists a Hölder continuous cocycle $\psi_+ : \mathsf{P}_+(\mathfrak{H}) \longrightarrow \mathbb{R}$ such that $\psi_+(\mathsf{P}_+(g)) = \psi'(g)$ for every $g \in \mathfrak{H}$.

Every element of $\mathsf{P}_+(\mathfrak{H})$ is equal to a product $\mathsf{P}_+(s_1) \cdots \mathsf{P}_+(s_n)$ where $s_i \in \mathcal{S} \cup \mathcal{S}^{-1}$. Let us define

$$\psi_+(\mathsf{P}_+(s_1) \cdots \mathsf{P}_+(s_n)) = \psi'(s_1) + \cdots + \psi'(s_n).$$

We have to show that ψ_+ is well defined. It is enough to show that if $\mathsf{P}_+(s_1) \cdots \mathsf{P}_+(s_n)$ is a unit, then $\psi'(s_1) + \cdots + \psi'(s_n) = 0$.

By [21, Lemma 6.7] there exist sequences $g_i, h_i, r_i, r_i' \in \mathfrak{H}$ such that $s_i = g_{i-1}^{-1} r_i g_i$, $\mathsf{P}_+(r_i) = \mathsf{P}_+(r_i')$, and $r_1' \cdots r_n' g_n$ is composable. Then

$$\mathsf{P}_+(s_1) \cdots \mathsf{P}_+(s_n) = \mathsf{P}_+(g_0^{-1}) \mathsf{P}_+(r_1' \cdots r_n' g_n),$$

and since this product is a unit, we have $\mathsf{P}_+(g_0) = \mathsf{P}_+(r_1' \cdots r_n' g_n)$, hence $\psi'(g_0) = \psi'(r_1' \cdots r_n' g_n)$.

But then $\sum \psi'(s_i) = \sum \psi'(g_{i-1}^{-1} r_i g_i) = \psi'(g_0^{-1}) + \psi'(g_n) + \sum \psi'(r_i) = \psi'(g_0^{-1}) + \psi'(g_n) + \sum \psi'(r_i') = \psi'(g_0^{-1}) + \psi'(r_1' \cdots r_n' g_n) = 0$.

Hölder continuity of ψ_+ follows from Hölder continuity of ψ' and the fact that every element of $\mathsf{P}_+(\mathfrak{H})$ is a product $\mathsf{P}_+(g)\mathsf{P}_+(h)$ for $g, h \in \mathfrak{H}$. Uniqueness of ψ_+ is straightforward. □

If $\psi_+ : \mathsf{P}_+(\mathfrak{H}) \longrightarrow \mathbb{R}$ satisfies the conditions of the last proposition for a cocycle $\psi : \mathfrak{H} \longrightarrow \mathbb{R}$, then we say that ψ_+ is a *projection* of ψ.

Note that if $\psi : \mathfrak{G} \longrightarrow \mathbb{R}$ is an arbitrary Hölder continuous cocycle on a hyperbolic groupoid, then its lift $\widetilde{\psi}(\xi, g) = \psi(g)$ onto the geodesic flow $\partial\mathfrak{G} \rtimes \mathfrak{G}$ is Hölder continuous and $\psi = \widetilde{\psi}_+$ is projection of $\widetilde{\psi}$. According to Proposition 8.2 there is a projection $\psi^\top : \mathfrak{G}^\top \longrightarrow \mathbb{R}$ of $-\widetilde{\psi}$ onto $\mathsf{P}_-(\partial\mathfrak{G} \rtimes \mathfrak{G})$. We call $\psi^\top = \mathsf{P}_+(-\widetilde{\psi}) = \mathsf{P}_-(\widetilde{\psi})$ the *dual cocycle* for the cocycle ψ. The dual cocycle is Hölder continuous and is uniquely defined up to continuous equivalence.

In particular, we have the following corollary of Proposition 8.2.

Corollary 8.4. *Every Hölder continuous cocycle on a hyperbolic groupoid is dualizable.*

The following explicit description of the dual cocycle $\psi^\top : \mathfrak{d}\mathfrak{G}_x \longrightarrow \mathbb{R}$ follows directly from the proof of Proposition 8.2.

Proposition 8.5. *Let $\mathfrak{G}$ be a minimal hyperbolic groupoid, and let $\psi : \mathfrak{G} \longrightarrow \mathbb{R}$ be a Hölder continuous cocycle.*

Define a map $\rho : T_g \longrightarrow \mathbb{R}$ by

$$\rho(f) = \psi(R_g^h(f)) - \psi(f).$$

Then $\rho(f)$ converges uniformly to $\psi^\top(R_g^h, \xi)$ as $f \to \xi$.

Proof. Let $\xi \in \partial\mathfrak{G}_x$ be equal to $\ldots g_2 g_1 \cdot g$ for $g_i \in S$. Let $U_i \in \mathcal{S}$ be such that g_i is ϵ-contained in U_i. It follows from the proof of Proposition 8.2 that cocycle is given by

$$\psi^\top(R_g^h, \xi) = \lim_{n\to\infty} \psi(U_n \cdots U_1 h) - \psi(U_n \cdots U_1 g).$$

Note that $R_g^h(U_n \cdots U_1 g) = U_n \cdots U_1 h$, so we have

$$\psi^\top(R_g^h, \xi) = \lim_{n\to\infty} \psi(R_g^h(U_n \cdots U_1 g) - \psi(U_n \cdots U_1 g)).$$

Let $A \subset \mathfrak{G}|_X$ be as in Proposition 2.3. Then for every n there exists $a \in A$ such that $\overline{T_{ag_n \cdots g_1 g}}$ is a neighborhood of ξ. Since ξ is an internal point of $\mathcal{T}_g$, for all n big enough we have $\overline{T}_{ag_n \cdots g_1 g} \subset \overline{T}_g$. Let $\mathcal{A}$ be a finite covering of A by bi-Lipschitz elements of $\widetilde{\mathfrak{G}}$. Let $\zeta = \ldots h_2 h_1 a g_n \cdots g_1 g \in \overline{T}_{ag_n \cdots g_1 g}$ (where the sequence h_i is finite or infinite), and suppose that $U \in \mathcal{A}$ and $V_i \in \mathcal{S}$ are such that a and h_i are ϵ-contained in U and V_i. Then, by Lemma 3.1, we have

$$R_g^h(\zeta) = \ldots V_2 V_1 U U_n \cdots U_1 h.$$

Since ψ is Hölder, we may assume that $\mathcal{S}$ is such that there exist constants c_1 and p such that $|\psi(V_i, x) - \psi(V_i, y)| \le c_1 |x - y|^p$ for all $V_i \in \mathcal{S}$, and $x, y \in \mathsf{o}(V_i)$.

Since V_i and U_i are contracting, and the elements of $\mathcal{A}$ are bi-Lipschitz, there exist $c > 1$ and $\lambda \in (0,1)$ such that

$$\begin{aligned}
&\big|\psi(U_n \cdots U_1 h) - \psi(U_n \cdots U_1 g) - \\
&\qquad (\psi(V_m \cdots V_1 U U_n \cdots U_1 h) - \psi(V_m \cdots V_1 U U_n \cdots U_1 g))\big| = \\
&\qquad |\psi(V_m \cdots V_1 U, \mathsf{t}(U_n \cdots U_1 h)) - \psi(V_m \cdots V_1 U, \mathsf{t}(U_n \cdots U_1 g))| \le \\
&\sum_{i=1}^{m} |\psi(V_i, \mathsf{t}(V_{i-1} \cdots V_1 U U_n \cdots U_1 h)) - \psi(V_i, \mathsf{t}(V_{i-1} \cdots V_1 U U_n \cdots U_1 g))| \le \\
&\qquad\qquad \sum_{i=1}^{m} c\lambda^{n+i} |\mathsf{t}(h) - \mathsf{t}(g)| \le \frac{c|\mathsf{t}(h) - \mathsf{t}(g)|}{1-\lambda} \lambda^n.
\end{aligned}$$

It follows that $\psi(R_g^h(f)) - \psi(f)$ uniformly converges to $\psi^\top(R_g^h, \xi)$ when $f \to \xi$. □

Example 8.1. Dual cocycle to the cocycle $\nu(F, z) = -\ln|F'(z)|$, where F belongs to the pseudogroup generated by a complex rational function, was studied in [14, Section 3.4].

8.3. **Conformal measures.**

Definition 8.5. Let $\nu : \mathfrak{G} \longrightarrow \mathbb{R}$ be a Hölder continuous Busemann cocycle on a hyperbolic groupoid. A Radon measure μ on $\mathfrak{G}^{(0)}$ is ν-conformal if

$$\rho_\mu(g) = e^{-\beta\nu(g)}$$

for every $g \in \mathfrak{G}$, where $\beta = h(\nu)$.

Proposition 8.6. *Suppose that $\nu_1 : \mathfrak{G}_1 \longrightarrow \mathbb{R}$ and $\nu_2 : \mathfrak{G}_2 \longrightarrow \mathbb{R}$ be continuously equivalent continuous cocycles. If there exists a ν_1-conformal measure on $\mathfrak{G}_1^{(0)}$, then there exists a ν_2-conformal measure on $\mathfrak{G}_2^{(0)}$.*

Proof. Suppose that the Radon-Nicodim derivative of μ is $e^{-\beta\nu}$ and let ν_1 be cohomologous to ν. If φ is the corresponding function such that $\nu_1(g) = \nu(g) + \varphi(\mathsf{t}(g)) - \varphi(\mathsf{o}(g))$, then the measure μ_1 given by $e^{\varphi(x)}\, d\mu(x)$ satisfies $\rho_{\mu_1}(g) = e^{-\beta\nu_1(g)}$.

It remains to prove that a ν-conformal measure on a hyperbolic groupoid $(\mathfrak{G}, \nu)$ exists if and only if it exists for its localization.

Let $X_0 \subset \mathfrak{G}^{(0)}$ be an open subset, and let μ_0 be a ν-conformal measure on X_0. Repeating the proof of Proposition 7.1 for the case of a conformal measure, we note that we get strict equalities everywhere instead of estimates. Consequently, conformal measures on open subsets are uniquely extended to conformal measures on the whole unit space. In the other direction, a conformal measure on $\mathfrak{G}^{(0)}$ restricted to an open subset X_0 is conformal with respect to the restriction of the groupoid. This implies immediately that a localization of $\mathfrak{G}$ has a conformal measure if and only if $\mathfrak{G}$ has a conformal measure. □

Theorem 8.7. *Let $(\mathfrak{G}, \nu)$ be a minimal hyperbolic groupoid graded by a Hölder continuous Busemann cocycle. Then there exists a unique, up to a multiplicative constant, ν-conformal measure on $\mathfrak{G}^{(0)}$.*

Proof. As usual, we will prove the theorem for the groupoid $(\mathfrak{G}^\top, \nu^\top)$, and then use duality. Let $X \subset \mathfrak{G}^{(0)}$ be a compact topological transversal, and let $\phi : \mathfrak{G}^{(0)} \longrightarrow \mathbb{R}$

be a non-negative continuous function with non-empty compact support that is a subset of X. Let $\beta = h(\mathfrak{G}, \nu)$. Define for $s > \beta$ a measure μ_s on $\mathfrak{G}_x$ by the equality

$$\int f \, d\mu_s = (1 - e^{\beta - s}) \sum_{g \in \mathfrak{G}_x} f(g)\phi(\mathsf{t}(g))e^{-s\nu(g)}. \tag{12}$$

Note that support of μ_s is a subset of $\mathfrak{G}_x^X$, since $\phi(\mathsf{t}(g)) = 0$ for $g \notin \mathfrak{G}_x^X$.

The same arguments as in Proposition 7.7 show that there exists a sequence $s_k \to \beta+$ such that μ_{s_k} weakly converge to a measure μ supported on $\partial\mathfrak{G}_x$.

Let us show that μ is $\nu^\top$-conformal. Let S and $\mathcal{S}$ satisfy the conditions of Proposition 2.2. Consider a germ (R_h^g, ξ) of the transformation $R_h^g : \overline{T}_h \longrightarrow \mathfrak{G}_x \cup \partial\mathfrak{G}_x$, where ξ is an internal point of $\mathcal{T}_h$, and $g, h \in \mathfrak{G}_x$ are such that $\mathsf{t}(g)$ and $\mathsf{t}(h)$ are sufficiently close to each other.

Recall that ϕ is uniformly continuous on $\mathfrak{G}^{(0)}$, f is continuous on $\overline{\mathfrak{G}_x^X}$, and there exist $c > 0$ and $\lambda \in (0, 1)$ such that $|\mathsf{t}(R_h^g(r)) - \mathsf{t}(r)| < c\lambda^{\nu(r)}$ for all $r \in T_h$. Let $C \subset \overline{\mathfrak{G}_x^X}$ be a compact neighborhood of ξ, and denote $C_0 = C \cap \mathfrak{G}_x$. We have

$$\frac{\int_{R_h^g(C)} f(r) \, d\mu_s(r)}{\int_C f(R_h^g(r)) \, d\mu_s(r)} = \frac{\sum_{s \in C_0} f(R_h^g(r))\phi(\mathsf{t}(R_h^g(r)))e^{-s\nu(R_h^g(r))}}{\sum_{s \in C_0} f(R_h^g(r))\phi(\mathsf{t}(r))e^{-s\nu(r)}}.$$

As we make the neighborhood C converge to ξ, the difference $|\phi(\mathsf{t}(R_h^g(r)) - \phi(\mathsf{t}(r))|$ uniformly converges to 0, while the difference $\nu(R_h^g(r)) - \nu(r)$ uniformly converges to $\nu^\top(R_h^g, \xi)$, by Proposition 8.5. The functions ϕ and f are bounded, the series $\sum_{r \in C} e^{-s\nu(r)}$ converges and has an upper bound not depending on C (see Lemma 7.6). Consequently,

$$\frac{\int_{R_h^g(C)} f(r) \, d\mu_s(r)}{\int_C f(R_h^g(r)) \, d\mu_s(r)} \to e^{-s\nu^\top(R_h^g, \xi)}$$

as C converges to ξ. It follows that μ is $\nu^\top$-conformal.

It remains to prove uniqueness of a conformal measure. It is enough to prove uniqueness of a conformal measure on $\partial\mathfrak{G}_x$ for some $x \in \mathfrak{G}^{(0)}$. Assume that S satisfies the conditions of Proposition 6.2. Let $|\xi - \zeta|$ be a metric on $\partial\mathfrak{G}_x$ of exponent α associated with the cocycle ν. We will denote by $B(\xi, r)$ the ball of radius r with center in ξ in $\partial\mathfrak{G}_x$.

Fix a number δ_0 that is small enough to satisfy the conditions of Lemma 3.1.

By Lemma 6.5, there exists a constant $N > 0$ such that for every $h_1, h_2 \in \mathfrak{G}_x$ there a transformation of the form $R_{h_1}^{gh_2}$ such that $R_{h_1}^{gh_2}(\mathcal{T}_{h_1}) \subset \mathcal{T}_{h_2}$, $0 < \nu(g) < N$, and $|\mathsf{t}(gh_2) - \mathsf{t}(h_1)| < \delta_0$. For every $\xi_1 \in \partial\mathfrak{G}_x$ there exists g_1 such that $B(\xi_1, 1) \subset \mathcal{T}_{g_1}$ (the proof is similar to the proof of [21, Proposition 4.9] (see also Proposition 2.3 of our paper).

It follows now from Propositions 6.2 that for all $r_1, r_2 > 0$ and $\xi_1, \xi_2 \in \partial\mathfrak{G}_x$ there exists a map of the form $R_{g_1}^{g_2}$ such that $R_{g_1}^{g_2}(\overline{B(\xi_1, r_1)}) \subset B(\xi_2, r_2)$, $\nu(g_2) - \nu(g_1)$ differs from $-\frac{\ln r_2 - \ln r_1}{\alpha}$ by a uniformly bounded constant, and $|\mathsf{t}(g_1) - \mathsf{t}(g_2)| < \delta_0$.

Fix $r_1 \in (0, 1)$, $\xi_1 \in \partial\mathfrak{G}_x$, and choose for every $r_2 \in (0, 1)$ and $\xi_2 \in \partial\mathfrak{G}_x$ a transformation $R_{g_1}^{g_2}$, and denote $V_{\xi_2, r_2} = \{\xi_2\} \cup R_{g_1}^{g_2}(B(\xi_1, r_1))$. Let $\mathcal{V}_{\xi_1, r_1}$ be the set of all sets of the form $V_{\xi, r}$ for $\xi \in \partial\mathfrak{G}_x$ and $r \in (0, 1)$. Then $\mathcal{V}_{\xi_1, r_1}$ is a covering of $\partial\mathfrak{G}_x$ by closed sets.

It follows from Proposition 8.5 and the fact that elements of $\mathcal{S}$ are λ-contractions for some fixed λ, that there exists a constant L depending only on $\mathcal{S}$ such that for

any germ $(R_{g_1}^{g_2}, \zeta)$ we have

$$|\nu^\top(R_{g_1}^{g_2}, \zeta) - (\nu(g_2) - \nu(g_1))| < L|\mathsf{t}(g_1) - \mathsf{t}(g_2)| \le L\delta_0.$$

By conformality of μ_1 and μ_2 we get

$$\frac{\mu_1(V_{\xi,r})}{\mu_2(V_{\xi,r})} = \frac{\int_{V_{\xi,r}} e^{-\beta\nu^\top(R_{g_1}^{g_2},\zeta)}\, d\mu_1(\zeta)}{\int_{V_{\xi,r}} e^{-\beta\nu^\top(R_{g_1}^{g_2},\zeta)}\, d\mu_2(\zeta)} \le \frac{e^{-\beta(\nu(g_2)-\nu(g_1))}e^{L\beta\delta_0}\cdot\mu_1(B(\xi_1, r_1))}{e^{-\beta(\nu(g_2)-\nu(g_1))}e^{-L\beta\delta_0}\cdot\mu_2(B(\xi_1, r_1))} = e^{2L\beta\delta_0}\frac{\mu_1(B(\xi_1, r_1))}{\mu_2(B(\xi_1, r_1))}.$$

We also conclude that there exist positive constants c_1, c_2 such that

$$c_1 e^{-\beta(\nu(g_2)-\nu(g_1))} \le \mu_i(R_{g_1}^{g_2}(\mathcal{T}_{g_1})) \le c_2 e^{-\beta(\nu(g_2)-\nu(g_1))},$$

for all transformations $R_{g_1}^{g_2}$ and for all $i = 1, 2$. It follows, by Proposition 7.4, that there exist constants c_3, c_4 such that

$$c_3 e^{\beta\ln r/\alpha} = c_3 r^{\beta/\alpha} \le \mu_i(V_{\xi,r}) \le c_4 e^{\beta\ln r/\alpha} = c_4 r^{\beta/\alpha}.$$

We will use a version of Vitali's covering theorem given in [5, Theorem 2.8.7]. Fix a constant $\tau > 1$ and denote for a set $V_{\xi,r} \in \mathcal{V}_{\xi_1,r_1}$ by $\widehat{V}_{\xi,r}$ the union of all the set of the form $V_{\zeta,s} \in \mathcal{V}_{\xi_1,r_1}$ such that $V_{\zeta,s} \cap V_{\xi,r}$ is non-empty, and $s \le \tau r$. Then $\widehat{V}_{\xi,r} \subset B(\xi, r + 2\tau r)$, since diameter of $V_{\zeta,s}$ is not greater than $2s \le 2\tau r$. It follows then from Proposition 7.4 that there exists a constant $c_5 > 0$ such that

$$\mu_i(\widehat{V}_{\xi,r}) < c_5(1 + 2\tau)^{\beta/\alpha} r^{\beta/\alpha} \le \frac{c_5(1 + 2\tau)^{\beta/\alpha}}{c_3}\mu_i(V_{\xi,r}).$$

Consequently, the conditions of [5, Theorem 2.8.7] are satisfied for the covering $\mathcal{V}_{\xi_1,r_1}$ of $\partial\mathfrak{G}_x$. Consequently, for any open subset $W \subset \partial\mathfrak{G}_x$ there exists a set $\mathcal{W}$ of pairwise disjoint elements of $\mathcal{V}_{\xi_1,r_1}$ such that $\bigcup\mathcal{W} \subset W$ and $\mu_i(W \setminus \bigcup\mathcal{W}) = 0$.

It follows from Proposition 7.4 that μ_i are doubling measures (see [12, Section 1.4]), hence they satisfy Lebesgue's differentiation theorem [12, Theorem 1.8]:

$$\lim_{r\to 0} \frac{1}{\mu_i(B(\xi, r))}\int_{B(\xi,r)} f\, d\mu_i = f(\xi) \tag{13}$$

for almost all ξ and for all locally integrable functions f.

The measures μ_1, μ_2 are mutually absolutely continuous by Corollary 7.5. The Radon-Nicodim derivative $d\mu_1/d\mu_2$ is constant on $\partial\mathfrak{G}^\top$-orbits, by conformality. Suppose that $d\mu_1/d\mu_2$ is not constant on $\partial\mathfrak{G}_x$. Then there exist $m_1 < m_2$ and sets of non-zero measure $A_1, A_2 \subset \partial\mathfrak{G}_x$ such that $d\mu_1/d\mu_2$ is less than m_1 on A_1 and bigger than m_2 on A_2. There exists $\xi_1 \in A_1$ such that

$$\lim_{r\to 0}\frac{\mu_1(B(\xi_1, r))}{\mu_2(B(\xi_1, r))} = \lim_{r\to 0}\frac{1}{\mu_2(B(\xi_1, r))}\int_{B(\xi_1,r)}\frac{d\mu_1}{d\mu_2}\, d\mu_2 = \frac{d\mu_1}{d\mu_2}(\xi_1) < m_1.$$

It follows for every $\epsilon > 0$ there exists r_1 such that $\frac{\mu_1(B(\xi_1,r))}{\mu_2(B(\xi_1,r))}$ is less than $m_1 + \epsilon$ for all $r \ge r_1$. Consider then the covering $\mathcal{V}_{\xi_1,r_1}$. For every $V \in \mathcal{V}_{\xi_1,r_1}$ we have $\frac{\mu_1(V)}{\mu_2(V)} < e^{2L\delta_0}(m_1 + \epsilon)$. Since every open subset of $\partial\mathfrak{G}_x$ can be represented as a countable union of disjoint elements of $\mathcal{V}_{\xi_1,r_1}$ and a zero-set, for every open set $W \subset \partial\mathfrak{G}_x$ we have $\frac{\mu_1(W)}{\mu_2(W)} \le e^{2L\delta_0}(m_1 + \epsilon)$. But ϵ and δ_0 can be made arbitrarily

small. Consequently, $\frac{\mu_1(W)}{\mu_2(W)} \le m_1$ for all open sets W, which is a contradiction with the inequalities $\lim_{r\to 0} \frac{\mu_1(B(\xi_2,r))}{\mu_2(B(\xi_2,r))} > m_2 > m_1$. □

Example 8.2. Let $f(z) \in \mathbb{C}(z)$ be a hyperbolic rational function, i.e., a complex rational function expanding on a neighborhood of its Julia set J_f. Then $f : J_f \longrightarrow J_f$ is a local homeomorphism, and the pseudogroup $\widetilde{\mathfrak{F}}$ generated by it is hyperbolic. We have seen (in Section 5) that a Busemann cocycle $\nu : \mathfrak{F} \longrightarrow \mathbb{Z}$ is given by the formula

$$\nu((f^n, x)^{-1} \cdot (f^m, y)) = n - m.$$

Note that ν is locally constant, hence Hölder continuous.

The conformal measure associated with ν is the weak limit of uniform distributions μ_n on the sets $f^{-n}(z_0)$ for any fixed $z_0 \in J_f$. It is also the measure of maximal entropy of the dynamical system (f, J_f), and is known (in the general setting of not necessarily hyperbolic functions) as Brolin-Lyubich measure [15].

The inverse limit $\mathcal{S}$ of the constant sequence of maps $f : J_f \longrightarrow J_f$ together with the homeomorphism on it induced by f is a Smale space called the *natural extension* of f. The groupoid $\mathfrak{F}$ is projection of this Smale space onto the unstable direction of the natural local product structure. The properties of the natural extensions (also in general, not only in the hyperbolic case), including their measure theory were studied in [16, 14].

The cocycle

$$\nu_1(F, z) = -\ln |F'(z)|$$

is another natural Busemann cocycle on $\mathfrak{F}$. Restriction onto J_f of the usual metric on $\mathbb{C}$ is a hyperbolic metric of exponent 1 associated with ν_1. Note that ν_1 is smooth, hence Hölder continuous. Measures conformal with respect to the cocycle ν_1 where defined for any complex rational function by D. Sullivan in [28]. It would be interesting to extend theory of hyperbolic groupoids to a more general setting, so that it will include all rational functions acting on the Julia set and all Kleinian groups acting on the limit set, see [26] (and not only geometrically finite groups without parabolic elements, as it is now).

The ν_1-conformal measure is, by Corollary 7.5, equivalent to the Hausdorff measure on the Julia set. In particular, the Hausdorff dimension of the Julia set is equal to the critical exponent of the series

$$\sum_{n\ge 0} \sum_{z\in f^{-n}(z_0)} e^{-s\ln |(f^{\circ n})'(z)|} = \sum_{n\ge 0} \sum_{z\in f^{-n}(z_0)} |(f^{\circ n})'(z)|^{-s}.$$

These results (existence of the conformal measure and formula for the Hausdorff dimension) are a partial case of [18, Theorem 1.2] due to C. McMullen.

8.4. **Invariant measure on the flow.** Let $(\mathfrak{G}, \nu)$ be a hyperbolic groupoid with a Hölder continuous Busemann cocycle. Suppose that $\mathfrak{G}^{(0)}$ is a disjoint union of rectangles.

Let ν_i for $i = +, -$ be cocycles cohomologous to ν such that $\mathsf{P}_i(\nu_i)$ are well defined on $\mathsf{P}_i(\mathfrak{G})$. Let $\phi_i : \mathfrak{G}^{(0)} \longrightarrow \mathbb{R}$ be such that

$$\nu(g) - \nu_i(g) = \phi_i(\mathsf{t}(g)) - \phi_i(\mathsf{o}(g)).$$

$$\nu_i(g) = \nu(g) - \phi_i(\mathsf{t}(g)) + \phi_i(\mathsf{o}(g)).$$

Consider then on each rectangle the direct product $\mu_+ \times \mu_-$ of the conformal measures defined by projections of ν_i and $-\nu_i$ onto the corresponding directions. We have

$$\rho_{\mu_+\times\mu_-}(g) = \exp(-\beta\nu_+(g) + \beta\nu_-(g)) =$$
$$\exp(-\beta(\nu(g) - \phi_+(\mathsf{t}(g)) + \phi_+(\mathsf{o}(g)) - \nu(g) + \phi_-(\mathsf{t}(g)) - \phi_-(\mathsf{o}(g)))) =$$
$$\exp(-\beta((\phi_-(\mathsf{t}(g)) - \phi_+(\mathsf{t}(g))) - (\phi_-(\mathsf{o}(g)) - \phi_+(\mathsf{t}(g))))).$$

It follows that the measure μ on $\mathfrak{G}^{(0)}$ given by

$$\int f(x)\, d\mu(x) = \int f(x) e^{-\beta(\phi_-(x) - \phi_+(x))}\, d\mu_+ \times \mu_-(x)$$

is invariant with respect to $\mathfrak{G}$.

We can extend this invariant measure to any groupoid equivalent to $\mathfrak{G}$, using the same methods as in Propositions 7.1 and 8.6.

Example 8.3. In the case when the cocycle $\nu : \mathfrak{G} \longrightarrow \mathbb{R}$ has values in $\mathbb{Z}$, the Smale quasi-flow $\mathfrak{G}$ is equivalent to a *Smale (orbi)space*. The corresponding invariant measure is the classical Bowen measure, see [1, 25], which is usually constructed using Markov partitions.

In general, for a continuous cocycle $\nu : \mathfrak{G} \longrightarrow \mathbb{R}$ on a Smale quasi-flow, the groupoid $\mathfrak{G}$ is equivalent to a *Smale flow* on an orbispace (see Proposition 8.1) and the constructed measure is a direct generalization of the Bowen-Margulis measure for Anosov flows, constructed in [17] and [2]. This follows from the scaling properties of the corresponding measures on the stable and unstable foliations, invariance under holonomies (i.e., definition of a ν-conformal measure), and the uniqueness statement of Theorem 8.7. See also [3].

The fact that Bowen-Margulis measure comes from the Hausdorff measures associated with natural metrics on the stable and unstable leafs (which follows now from Corollary 7.5) was proved by B. Hasselblatt [11]. In the case of a geodesic flow on a negatively curved manifold this was proved by U. Hamenstädt [10].

Note that the geodesic flow on a negatively curved compact manifold M is equivalent (as a topological groupoid) to the action of the fundamental group $\pi_1(M)$ on the square $\partial\widetilde{\mathcal{M}} \times \partial\widetilde{\mathcal{M}}$ of the ideal boundary of the universal covering of M, minus the diagonal. It follows that the groupoid generated by the geodesic flow is equivalent to the geodesic flow $\partial\mathfrak{G} \rtimes \mathfrak{G}$ of the hyperbolic groupoid $\mathfrak{G}$ of the action of the fundamental group $\pi_1(M)$ on its Gromov boundary (equivalently, on $\partial\widetilde{\mathcal{M}}$). We obtain in this way the well known fact that the Bowen-Margulis measure associated with a geodesic flow on a negatively curved compact manifold M can be obtained from the Patterson-Sullivan measure on $\partial\widetilde{\mathcal{M}}$. See the paper of D. Sullivan [27], for the constant curvature case, and the paper of V. Kaimanovich [13] for the general case. Note that it is also shown in the latter paper that the Patterson-Sullivan measures are Hausdorff measures of naturally defined metrics. The paper [13] also considers measures and metrics arising from different choices of the cocycle.

Conformal measures on the groupoid $\mathfrak{F}$ generated by a complex rational function and on its dual $\mathfrak{F}^\top$ are studied in the monograph [14].

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